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Asset Bubbles, Entrepreneurial Risks, and Economic Growth

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Abstract

Entrepreneurs are exposed to large uninsured risks. The risks may discourage them from creating productive assets. This may generate a productive asset shortage and stimulate speculative demand for bubbles. We introduce within-period entrepreneurial risks into a textbook growth model with infinitely lived agents. In our model, entrepreneurs face no credit constraints. If the degree of entrepreneurial risks is in the middle range, bubbles are likely to emerge. If the degree of entrepreneurial risks is high, bubbles promote growth because of the wealth effect. Otherwise, bubbles lower growth. The effect of the collapse of bubbles also depends on the degree of the risks. Moreover, asset bubbles amplify fundamental shocks. (108 words)

Keywords: asset bubbles, idiosyncratic risks, amplification, growth effect, welfare analysis.

JEL classification numbers: E21, E23, E44, G01, G11

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1 Introduction

This study explores the interactions between asset bubbles and uninsurable entrepreneurial risks in an infinitely lived agents model. Entrepreneurial risks may give rise to bubbles. Evidence shows that entrepreneurs are exposed to large uninsured idiosyncratic risks (Heaton and Lucas, 2000; Moskowitz and Vissing-Jørgenssen, 2002; Gentry and Hubbard, 2004). As Caggese (2012) and Michelacci and Schivardi (2013) show, uninsured risks hinder entrepreneurial activities and suppress the creation of productive assets, such as new businesses and production facilities. This causes a shortage of supply of productive assets. As Caballero (2006) suggests, the shortage increases the price of productive assets and lowers the return on productive assets, which may stimulate speculative demand for bubbly assets (also, see chapters in Baldwin and Teulings, 2014).[1]

In addition, asset bubbles may affect the risk-taking behavior of individuals. Some evidence suggests that wealthier individuals may be more willing to take risks than those with less wealth (Guiso and Paiella, 2008; Dohmen *et al.*, 2011; Guiso and Sodini, 2013; Haushofer and Fehr, 2014). This suggests that a rise and burst of asset bubbles, causing large fluctuations in wealth, may affect entrepreneurs' decisions on risky activities.

The interaction between bubbles and entrepreneurial risks potentially has substantial impacts on the aggregate economy. As Michelacci and Schivardi (2013) empirically show, entrepreneurial risks significantly affect the macroeconomy and aggregate growth. In addition, as economic history has repeatedly shown, large depressing effects follow the burst of asset bubbles. This study examines how bubbles and entrepreneurial risks interact and then the macroeconomic effects of the interaction.

Existing theoretical studies have focused on entrepreneurial risks and asset bubbles separately. Authors like Angelotos and Calvet (2006) and Angelotos (2007) examine the effects of uninsured entrepreneurial risks on capital accumulation. However, they do not consider bubbles. Most studies on rational bubbles care less about uncertainty regarding the outcomes of entrepreneurial activities, although some studies consider productivity shocks. Recent studies on rational bubbles emphasize the roles of credit constraints in infinitely-lived agents economies (Kunieda and Shibata, 2016; Hirano and Yanagawa, 2017; Miao and Wang, 2018).[2] As in Bewley (1980), the uninsured idiosyncratic shocks make borrowing constraints occasionally binding and then infinitely lived agents hold bubbly assets to ease binding borrowing constraints. Moreover, bubbles reallocate resources from lenders with low productivity to borrowers with high productivity and thus may boost long-run growth. However, these authors commonly assume that agents observe the realization of the shocks *before* they make production or investment decisions and portfolio choice. In other words, there is no within-period uncertainty on entrepreneurial activities.

This study introduces uninsured risks into a textbook AK model with infinitely-lived agents. Even if we use a neoclassical production function, our main results are unaffected. Risk-averse entrepreneurs earn income from asset holdings and creation of new productive assets. They hold both productive and bubbly assets. The creation of the new productive

[1] In addition, the shortage of assets may cause various problems, including secular stagnation. See Baldwin and Teulings (2014).

[2] Fahri and Tirole (2011) and Martin and Ventura (2012) introduce credit constraints into overlapping-generations economies and show that bubbles may stimulate growth in the long run.

assets is subject to idiosyncratic shocks, reflecting the risks of opening new businesses, developing new technologies, and so on. We call the productive assets *capital*. The shocks are realized only *after* entrepreneurs implement production and make portfolio choice. Thus, entrepreneurs are unable to know the exact outcomes of their activities in advance. In this sense, there is within-period uncertainty. Since entrepreneurs have ex-ante identical productivity, different entrepreneurs do not lend to and borrow from each other. Hence, credit constraints do not matter. Even without borrowing constraints, bubbles may arise and promote growth in an infinitely-lived agents economy. Interestingly, bubbles amplify fundamental shocks.

Since entrepreneurs choose their portfolios before the risk realizes, the realization has no impact on their portfolio choices between productive and bubbly assets. In addition, since borrowing constraints do not matter, entrepreneurs do not hold bubbles to ease borrowing constraints. This contrasts with the above-mentioned Bewley-type models, where borrowing constraints may or may not bind depending on the realization of the shock, and agents sell bubbly assets when they face a binding borrowing constraint. These features enable us to focus on the speculative aspects of bubbles.

Moreover, the absence of credit constraints simplifies our model. If we remove the entrepreneurial risks, our model reduces to a textbook AK (or neoclassical growth) model. This simplicity enables us to derive all the results analytically. Therefore, we can easily compare our results with those obtained in standard macroeconomic models. Our rational bubble model could be applied to a wide range of economic analyses such as business cycle and policy analysis. The construction of such a simple model is one of our contributions.

We show that bubbles may emerge, depending on the degree of the entrepreneurial risks, which we denote as σ. If there are no risks ($\sigma = 0$), infinitely lived agents do not hold asset bubbles. However, in the presence of uninsured risks ($\sigma > 0$), bubbles may emerge. Faced with uninsured risks, entrepreneurs reduce production of productive assets. The price of productive assets increases and the rate of return on holding productive assets decreases. Since bubbly assets yield a high return, entrepreneurs hold bubbly assets for a speculative purpose. However, if the entrepreneurial risks are extremely large (σ is too large), the economy grows too slowly, which cannot sustain the expansion of bubbles. Thus, if the degree of the risks σ is in the middle range, asset bubbles arise.

Bubbles affect entrepreneurs' productive asset creation and thus long-run growth. Whether bubbles promote growth depends on the degree of the risks σ. As in existing models, bubbles have a negative growth effect (*e.g.*, Tirole, 1985). With uninsured risks ($\sigma > 0$), bubbles have a positive growth effect. Asset bubbles make entrepreneurs wealthy. The wealthy entrepreneurs undertake large-scale production in spite of large risks, which has a positive growth effect. On the condition that bubbles exist, if entrepreneurs face relatively large risks, the positive effect dominates the negative one. Then, bubbles promote growth. By contrast, if entrepreneurs face relatively small risks, bubbles harm growth.

The effect of the collapse of bubbles also depends on the degree of the entrepreneurial risks σ. If entrepreneurs face relatively large risks, the collapse of bubbles depresses growth. In our model, a sunspot shock causes the collapse of bubbles. Without any changes in fundamentals, the burst of bubbles triggers a sudden and permanent change in growth.

Moreover, bubbles amplify the effects of changes in economic fundamentals. Consider a marginal change in the TFP of AK technology. In the presence of bubbles, a marginal increase in TFP promotes growth more than in the absence of bubbles. The intuition is simple. An increase in TFP enhances growth. Thus, the economy sustains a large size of bubbles. Large bubbles boost economic growth further (if entrepreneurs face large risks).

We also examine the welfare effects of bubbles. Risk-averse entrepreneurs suffer utility loss from risky capital production. Bubbles mitigate the utility loss, because bubbles make entrepreneurs wealthy and increases their tolerance to the risks. Thus, bubbles always improve the welfare of all entrepreneurs.

Related literature: Our study is related to Angelotos and Calvet (2006) and Angelotos (2007) who construct models with uninsured production risks. These authors show that uninsured risks depress capital accumulation. The structure of our model is similar to that of these models. However, they do not consider bubbles. We show that uninsured entrepreneurial risks depress capital production and hence lower the return on productive assets, which generates the demand for speculative bubbles. In addition, we show that bubbles may accelerate capital accumulation. Candian and Dmitriev (2020) focus on within-period entrepreneurial risks in a model without bubbles. They show that uninsured risks mitigate fundamental shocks. By contrast, we show that the uninsured risk gives rise to bubbles and the bubbles amplify fundamental shocks.

This study is also related to the literature on rational bubbles. In standard models with infinitely-lived agents, bubbles are often ruled out. Thus, the (two-period) overlapping-generations (OLG) framework has traditionally been used to study bubbles. In traditional models, such as Tirole (1985), bubbles crowd investment out and lower capital accumulation and output in the long run.[3] However, Fahri and Tirole (2011) show that, in the presence of credit constraints, bubbles may promote capital accumulation. Similarly, Martin and Ventura (2012) show that newly-created bubbles ease borrowing constraints and enhance capital accumulation. Unlike these studies, we develop an infinitely-lived agents model of rational bubbles without new bubble creation and credit constraints. A potential benefit of an infinitely-lived agents model is its suitability for quantitative analyses.

Bewley (1980) shows that agents hold bubbles (fiat money) in an endowment economy with infinitely-lived agents.[4] In his model, agents face a borrowing constraint that binds occasionally.[5] After income shocks hit agents, they decide whether to hold fiat money. Depending on the realizations of the income shock, borrowing constraints may or may not bind. If an agent faces a binding borrowing constraint, the agent sells bubbly assets to ease the constraint. If an agent's borrowing constraint is not binding, the agent purchases bubbly assets as self-insurance against a binding borrowing constraint in the subsequent periods.

Similarly, Kocherlakota (2009), Kunieda and Shibata (2016), Hirano and Yanagawa (2017), and Miao and Wang (2018) recently show that with occasionally binding credit constraints, bubbles may exist in production economies with infinitely lived agents. Moreover, these studies show that bubbles promote capital accumulation and long-run growth (under some conditions), because bubbles reallocate productive resources from unproductive agents to

[3]Grossman and Yanagawa (1993) and King and Ferguson (1993) find that asset bubbles retard long-run economic growth.

[4]Kocherlakota (1992) and Santos and Woodford (1997) also construct endowment economy models of rational bubbles in which infinitely-lived agents face borrowing constraints.

[5]As Aiyagari and McGrattan (1998) and Fahri and Tirole (2011) point out, occasionally binding borrowing constraints shorten agents' planning horizon and make infinitely-lived agents' behavior similar to that of OLG models.

financially-constrained productive agents.[6] However, as we mention earlier, there is no within-period uncertainty in these models, which is crucial in our model.

This study also shows that bubbles may arise in an infinitely lived agents economy and enhance growth. However, there are notable differences between these models with credit constraints and ours. First, we assume that entrepreneurs implement production and choose portfolio allocation between productive and bubbly assets before the realization of the entrepreneurial risk. Thus, the realization of the risk does not influence entrepreneurs' portfolio choice.

Second, since entrepreneurs in our model have ex-ante identical productivity, they do not lend to and borrow from each other and thus credit constraints do not matter. In the models with credit constraints, heterogeneous productivity and the occasionally binding credit constraints are essential for bubbles to emerge and promote long-run growth. We focus on a smaller set of ingredients by abstracting heterogeneous productivity and credit constraints. We show that these two factors may not be necessary for asset bubbles to emerge and promote growth.

Finally, the absence of credit constraints simplifies our model. We do not claim that our model is superior to the existing models of rational bubbles. However, the simplicity of our model enables us to conduct detailed analyses and thus to provide some new theoretical insights that are not explored in existing studies. For example, we show how bubbles amplify a technology shock, which is not formally addressed in the abovementioned studies. Moreover, since our simply model is quite close to a textbook model, our model is possibly applied to a wide range of economic analysis.

Aoki *et al.* (2014) also show that bubbles may arise in an infinitely lived agents model without borrowing constraints. In their model, the rate of return on holding bubbly assets is lower than that of holding productive assets. Thus, there is no speculative motive for holding bubbly assets. Instead, agents hold bubbly assets to diversify idiosyncratic risks.[7] In our model, entrepreneurs hold bubbly assets for speculative purposes. Moreover, in Aoki *et al.* (2014), bubbles always lower growth, regardless of the degree of the idiosyncratic risks.

The rest of the paper is organized as follows. Section 2 presents our model. Section 3 examines how bubbles emerge, affect growth and welfare, and amplify fundamental shocks. Concluding remarks are presented in Section 4.

2 A Simple AK Model

Time is continuous and runs from $t = 0$ to ∞. Entrepreneurs own productive assets and bubbly assets. We call the productive assets *capital*. A single general good is produced through an AK production technology using capital.[8] The asset bubbles collapse stochastically. As long as bubbles continue to exist, both the rates of return on holding capital and bubbles are deterministic. Entrepreneurs produce new capital using the general good, which is subject to

[6] A similar mechanism is at work in the OLG model of Martin and Ventura (2012). Woodford (1990) is an early study showing that bubbles promote capital investment. Olivier (2000) and Tanaka (2011) investigate how stock bubbles stimulate R&D activities.

[7] Kitagawa (1994) shows that agents demand bubbly assets as safe assets in an OLG model.

[8] If we employ a neoclassical production function, our main results are not affected. See Appendix O.

idiosyncratic risks. We interpret capital broadly. Appendix N presents an extended model in which production of productive assets includes setting up new businesses, opening branches, and developing new products and technologies, and these activities bear idiosyncratic risks.

2.1 General Good Sector

A single general good is used for both consumption and input of capital production. The general good is competitively produced by the following production technology:

$$Y_t = AK_t, \quad A > 0, \tag{1}$$

where Y_t and K_t denote the output and capital input, respectively. The general good is taken as a numeraire. We denote the rental rate of capital by q_t. Profit maximization yields

$$q_t = A. \tag{2}$$

2.2 Entrepreneurs

Preferences: There is a continuum of infinitely lived entrepreneurs whose measure is one. They are risk averse. Entrepreneur $i \in [0, 1]$ has the following expected lifetime utility:

$$U_{i,t} = E_t \int_t^\infty (\log c_{i,t})\, e^{-\rho(s-t)} ds, \tag{3}$$

where $c_{i,t}$ is entrepreneur i's consumption, $\rho > 0$ is the subjective discount rate, and E_t is an expectation operator conditional on time t information. We assume that

$$A > \rho. \tag{4}$$

Asset holdings and portfolio: Entrepreneurs own productive assets (capital) and bubbly assets. Let v_t and p_t be the capital price and bubbly asset price, respectively, at time t. Entrepreneur i holds $k_{i,t}$ units of capital and $b^n_{i,t}$ units of bubbly assets. His or her total asset holdings are given by

$$\omega_{i,t} = v_t k_{i,t} + p_t b^n_{i,t} = a_{i,t} + b_{i,t}, \tag{5}$$

where $a_{i,t} \equiv v_t k_{i,t}$ and $b_{i,t} \equiv p_t b^n_{i,t}$. We assume that $\omega_{i,0} > 0$ for all entrepreneurs. The total nominal supply of bubbly assets is constant at $M > 0$.

Entrepreneurs lend their capital to general good firms at the rental rate q and earn capital rental income. The rate of return on holding capital is given by

$$r_t dt \equiv \frac{qdt + dv_t - \delta v_t dt}{v_t},$$

where $\delta > 0$ is the capital depreciation rate.

As in Tirole (1985), a bubbly asset is an intrinsically useless asset with zero fundamental value. The free disposability of bubbly assets ensures $p_t \geq 0$. In a *bubbleless economy*, p_t is 0

(p_t=0). In a *bubbly economy*, p_t is strictly positive ($p_t > 0$) and the rate of return on holding bubbly assets is

$$\psi_t dt \equiv \frac{dp_t}{p_t}.$$

Bubbles may burst stochastically in the future. As in Weil (1987), once bubbles burst, they are never valued in the subsequent future. A sunspot shock triggers the burst of a bubble. Given that $p_t > 0$, p_{t+dt} remains strictly positive with probability $1-\mu dt$ ($\mu > 0$). Otherwise, we have $p_{t+dt} = 0$. A larger μ means riskier bubbles. All entrepreneurs know the value of μ. All the propositions in this study hold even if bubbles never burst $\mu = 0$.

Since the sunspot shock is the only aggregate shock, both r_t and ψ_t are deterministic as long as the bubbly economy prevails. Similarly, r_t is deterministic in a bubbleless economy.

Productive asset production and budget constraint: Productive asset (capital) production is irreversible and subject to uninsured risks. If entrepreneur i uses $I_{i,t}(\geq 0)$ units of the general good for a time period of length dt, then $dx_{i,t}$ units of capital are produced as follows:

$$dx_{i,t} = \phi I_{i,t}dt + \sigma I_{i,t}dW_{i,t}, \qquad \phi = 1, \quad \sigma > 0, \tag{6}$$

where $W_{i,t}$ is a standard Brownian motion. Its increment, $dW_{i,t}$, represents idiosyncratic capital production risks. $dW_{i,t}$ is independent and identically distributed across entrepreneurs. $dW_{i,t}$ is realized after entrepreneur i chooses $I_{i,t}$. In this sense, there is a within-period uncertainty in the time period of length dt. Parameters ϕ and σ are common to all entrepreneurs. We normalize $\phi = 1$. As in Angeletos and Calvet (2006), σ represents the degree of the idiosyncratic risks. With a large σ, entrepreneurs face large risks. Without uninsured risks $\sigma = 0$, our model reduces to a standard AK model. As mentioned at the beginning of this section, $dx_{i,t}$ includes starting new businesses, developing new technologies, and so on.

There is an aggregate market for productive assets. Entrepreneurs sell capital that they newly produce at the capital price v_t. Since the general good price is one and $\phi = 1$, entrepreneur i earns the following profits:

$$(v_t - 1)I_{i,t}dt + \sigma v_t I_{i,t}dW_{i,t}. \tag{7}$$

All entrepreneurs have the same ex-ante productivity $\phi(= 1)$ and learn shocks *after* output is realized. Thus, lending and borrowing among different entrepreneurs do not occur. Hence, borrowing constraints do not matter.

In the bubbly economy, the budget constraint faced by entrepreneur i (see Appendix A) is given by

$$d\omega_{i,t} = \left\{\left[r_t(1 - s_{i,t}) + \psi_t s_{i,t}\right]\omega_{i,t} - c_{i,t}\right\}dt + \left\{(v_t - 1)dt + \sigma v_t dW_{i,t}\right\}I_{i,t}, \tag{8}$$

where $s_{i,t} \equiv b_{i,t}/\omega_{i,t}$ is the portfolio weight of the bubbly assets, which captures the portfolio choice between the productive assets (capital) and the bubbly assets. In the bubbleless economy, we have $p_t = b_{i,t} = s_{i,t} = 0$ in (8).[9] We distinguish the income from productive

[9]Moll (2014) considers a productivity shock in a continuous-time growth model without bubbles, assuming that the shock is realized before the agents make production decisions. He considers a budget constraint like $d\omega_{i,t} = \{r_t\omega_{i,t} - c_{i,t} + (\phi_t v_t - 1)I_{i,t}\}dt$, where ϕ_t follows a stochastic process.

asset creation, $I_{i,t}[(v_t-1)dt+\sigma v_t dW_{i,t}]$, with the income from productive asset holdings, $r_t(1-s_{i,t})\omega_{i,t}$. The former is subjective to the idiosyncratic risk $\sigma dW_{i,t}$, whereas the latter bears no risks. Thus, $\sigma dW_{i,t}$ affects the entrepreneurs' decision on $I_{i,t}$, whereas $\sigma dW_{i,t}$ does not affect their portfolio choice $s_{i,t}$.

Utility maximization: Given $\omega_{i,0}>0$, entrepreneur i maximizes (3) subject to (5) and (8). At each moment of time, entrepreneur i chooses $c_{i,t}$, $s_{i,t}$, and $I_{i,t}$. We impose a non-negativity constraint, $k_{i,t}\geq 0$. The short sales constraint on the bubbly assets $b^n_{i,t}\geq 0$ must be satisfied. Households' optimization problem must also satisfy a no-Ponzi-game condition, $\lim_{T\to\infty}\omega_{i,T}e^{-\int_t^T r_v dv}\geq 0$. Appendix B shows that the behavior of entrepreneur i can be summarized as follows:

$$c_{i,t}=\rho\omega_{i,t}, \tag{9a}$$

$$I_{i,t}=\frac{v_t-1}{(\sigma v_t)^2}\omega_{i,t}, \qquad \sigma>0, \tag{9b}$$

$$s_{i,t}=s_t=\begin{cases}1-\frac{\mu}{\psi_t-r_t} & \text{in the bubbly economy } (p_t>0),\\ 0 & \text{in the bubbleless economy } (p_t=0),\end{cases} \tag{9c}$$

$$d\omega_{i,t}=\left[r_t(1-s_t)+\psi_t s_t+\left(\frac{v_t-1}{\sigma v_t}\right)^2-\rho\right]\omega_{i,t}dt+\left(\frac{v_t-1}{\sigma v_t}\right)\omega_{i,t}dW_{i,t}, \qquad \sigma>0. \tag{9d}$$

(9a) is a usual consumption function under a logarithmic utility function. We assume an inner solution for $I_{i,t}\geq 0$, which is satisfied in the equilibrium we consider. (9d) shows that $\omega_{i,t}$ follows a generized geometric Brownian motion, which ensures that $\omega_{i,t}>0$, because $\omega_{i,0}>0$ (see Example 4.4.8 on p.147-148 in Shreve 2004). The no-Ponzi game condition is satisfied. Besides, the transversality condition is satisfied as $\lim_{t\to\infty}E_t\left[\frac{\omega_{i,t}}{c_{i,t}}e^{-\rho t}\right]=\lim_{t\to\infty}\frac{1}{\rho}e^{-\rho t}=0$.

The degree of risk affects the creation of new productive assets (see (9b)). With $\sigma>0$, only if the capital price is higher than the marginal cost of capital production ($v_t>1$), entrepreneurs choose positive capital production ($I_{i,t}>0$). As σ increases, risk-averse entrepreneurs decrease capital production.

(9c) is also a standard result and shows entrepreneur i's portfolio decision between the productive assets (capital) and the bubbly assets. All entrepreneurs put the same portfolio weight on the bubbly assets, $s_{i,t}=s_t$. Given $\omega_{i,t}$, the entrepreneur allocates $\omega_{i,t}$ between $a_{i,t}$ and $b_{i,t}$ by considering the market returns of assets, r_t and ψ_t, and the bursting rate μ. Thus, the portfolio weight s_t depends only on r_t, ψ_t, and μ. Since the idiosyncratic shock does not have any direct effects on r_t and ψ_t, s_t does not depend on σ directly.

As to s_t, we mention the following four points. First, the short-sales constraint $s_t\geq 0$ never binds. Since the nominal supply of bubbly assets is strictly positive $M>0$, the fact $s_{i,t}=s_t$ means that all entrepreneurs hold a positive bubbly assets $b_{i,t}=s_t\omega_{i,t}>0$ in the bubbly economy. The short-sales constraint never binds and $s_t>0$ holds.[10] Similarly, since capital stock is always positive ($K_t>0$), we must have $v_t k_{i,t}=(1-s_t)\omega_t>0$ in equilibrium. Second, the fact that s_t does not depend on σ suggests that entrepreneurs hold bubbly assets not for the diversification of entrepreneurial risks, which contrasts with Kitagawa (1994) and

[10] Kocherlakota (1992) shows that if individuals borrow and lend, a short sales constraint $b^n_{i,t}\geq 0$ is needed for the existence of bubbles.

Aoki *et al.* (2014).[11] Third, the term $\psi_t - r_t$ in (9c) represents the risk premium on bubbles that is positive in equilibrium if $\mu > 0$ (see (18d)).[12] The large risk premium stimulates entrepreneurs' demand for bubbly assets. Finally, since entrepreneurs choose $s_{i,t}$ before $dW_{i,t}$ is realized, s_t does not depend on the shock $dW_{i,t}$. In models with credit constraint, the realization of the shocks affects the agents' portfolios (see Kocherlakota (2009), Kunieda and Shibata (2016), Hirano and Yanagawa (2017), and Miao and Wang (2018)). In these models, bubbles loosen the credit constraints. In our model, this mechanism is absent since the credit constraint does not matter.

The idiosyncratic nature of the production risks is essential for the trade of capital and the bubbly assets among entrepreneurs. As we show later, s_t is constant over time in the equilibrium. Then, $a_{i,t} = (1-s)\omega_{i,t}$ and $b_t = s\omega_{i,t}$ imply that $da_{i,t} = (1-s)d\omega_{i,t}$ and $db_{i,t} = sd\omega_{i,t}$. As (9d) shows, the idiosyncratic shock $dW_{i,t}$ generates heterogeneity in $d\omega_{i,t}$. This triggers the trade of productive assets (capital) $da_{i,t}$ and the bubbly assets $db_{i,t}$ among entrepreneurs. Entrepreneurs with $d\omega_{i,t} > 0$ buy both $a_{i,t}$ and $b_{i,t}$, whereas entrepreneurs with $d\omega_{i,t} < 0$ sell both. Notice that entrepreneurs choose s_t before the realization of $dW_{i,t}$.

2.3 Aggregation and Competitive Equilibrium

Let us define the following aggregate valuables, $C_t = \int_0^1 c_{i,t}di$, $I_t = \int_0^1 I_{i,t}di$, $K_t = \int_0^1 k_{i,t}di$, $b_t^n = \int_0^1 b_{i,t}^n di$, and $\omega_t = \int_0^1 \omega_{i,t}di$. Then, we have

$$\omega_t = v_t K_t + p_t b_t^n, \tag{10a}$$

$$C_t = \rho\omega_t, \tag{10b}$$

$$I_t = \frac{v_t - 1}{(\sigma v_t)^2}\omega_t, \qquad \sigma > 0. \tag{10c}$$

Because $I_{i,t}$ and $dW_{i,t}$ are independent and $dW_{i,t}$ follows a normal distribution with mean zero, we aggregate (6) as $dK_t \equiv \int_0^1 (dx_{i,t})di - \delta K_t dt = [I_t + \sigma \int_0^1 I_{i,t}di \int_0^1 (dW_{i,t})di - \delta K_t]dt = [I_t - \delta K_t]dt$.[13] The growth rate of the economy is given by

$$g_t = \frac{\dot{K}_t}{K_t} = \frac{I_t}{K_t} - \delta. \tag{11}$$

The total nominal supply of bubbly assets is constant at $M > 0$. The market for bubbly assets clears as $b_t^n = M$. The general good market clears as

$$Y_t = C_t + I_t. \tag{12}$$

[11] Kitagawa (1994) and Aoki *et al.* (2014) consider a budget constraint like $d\omega_{i,t} = (r_t dt + \sigma_k dW_{i,t}^k)a_{i,t} + (\psi_t b_{i,t} - c_{i,t})dt$, where $W_{i,t}^k$ is a standard Brownian motion. If $\sigma_k > 0$, the rate of return on holding capital, $r_t dt + \sigma_k dW_{i,t}^k$, is stochastic and idiosyncratic. Holding the bubbly asset, $b_{i,t}$, diversifies this capital holding risk. Thus, in their model, the portfolio weight on the bubbly assets depends on σ_k. This risk diversification motive is absent from our model.

[12] If $\mu = 0$, then $\psi_t = r_t$ holds in the bubbly economy. In this case, s_t is indeterminate at the individual entrepreneur level. However, this does not affect our main results.

[13] Because $dW_{i,t}$ has zero mean and is independent and identically distributed among entrepreneurs, $\int_0^1 (dW_{i,t})di = 0$ holds, owing to the law of large numbers. See Uhlig (1996).

For later use, let us define V_t and B_t as follows:

$$V_t \equiv \frac{1}{v_t} \quad \text{and} \quad B_t \equiv \frac{p_t M}{v_t K_t}. \tag{13}$$

V_t is the price of the general good in terms of capital and B_t is the value of bubbles relative to the value of capital. Both V_t and B_t are jump variables. We have $B_t > 0$ in the bubbly economy, whereas we have $B_t = 0$ in the bubbleless economy. Because $p_t M = s_t \omega_t$ holds from $b_{i,t} = s_t \omega_{i,t}$ and $b_t^n = M$, we have $s_t = B_t/(1 + B_t)$. Thus, $s_t \in (0, 1)$ holds in the bubbly economy ($B_t > 0$). A steady-state equilibrium is an equilibrium in which V_t and B_t are constant. In the steady state, g_t becomes constant, and K_t, C_t, and Y_t grow at the same rate. In the bubbly steady state, p_t also grows at the same rate as K_t.

2.4 Economy without Uninsured Entrepreneurial Risks: $\sigma = 0$

Without the uninsured entrepreneurial risks ($\sigma = 0$), our model reduces to a standard AK model and asset bubbles cannot exist, as shown in the following proposition.

Proposition 1 *Suppose that $\sigma = 0$ and (4) hold. (i) There exists a unique bubbleless equilibrium, where V_t, r_t, and g_t satisfy*

$$V_t = 1 \equiv V_{NR}, \quad r_t = A - \delta \equiv r_{NR}, \quad \text{and} \quad g_t = A - \rho - \delta \equiv g_{NR} \ \ (< r_{NR}). \tag{14}$$

Inequality (4) ensures that $I_t > 0$. (ii) There is no bubbly economy.

(Proof) See Appendix C.

3 Insured Risks($\sigma > 0$) and Bubbles

The uninsured risk $\sigma > 0$ gives rise to asset bubbles and examines how bubbles affect long-run growth. We first provide two equations that characterize the equilibrium.

Proposition 2 *Assume that $\sigma > 0$. In an equilibrium in which $I_t > 0$ holds, V_t and B_t satisfy*

$$A = \left[\frac{\rho}{V_t} + \frac{1 - V_t}{\sigma^2}\right](1 + B_t), \tag{15a}$$

$$\dot{B}_t = \left\{\mu(1 + B_t) + AV_t - \frac{1 - V_t}{\sigma^2}(1 + B_t)\right\} B_t. \tag{15b}$$

(Proof) See Appendix D.

(15a) comes from the general good market equilibrium condition (12). The left-hand side (LHS) shows the general good supply (Y_t/K_t), while the right-hand side (RHS) shows the general good demand ($(C_t + I_t)/K_t$). (15b) gives the dynamics of B_t.

3.1 Bubbleless Economy

In the bubbleless economy, in which $B_t = \dot{B}_t = 0$ holds, (15a) alone determines V_t. We prove the following proposition.

Proposition 3 *Assume that $\sigma > 0$. If and only if* (4) *holds, a unique bubbleless steady-state equilibrium exists such that $I_t > 0$ holds and V_t, r_t, and g_t satisfy*

$$V_t = V_L \quad (< V_{NR} \equiv 1), \tag{16a}$$

$$r_t = AV_L - \delta \equiv r_L \quad (< r_{NR}), \tag{16b}$$

$$g_t = \frac{1 - V_L}{\sigma^2} - \delta \equiv g_L \quad (< g_{NR}), \tag{16c}$$

where $V_L \in (\rho/A,\ 1)$ is a positive solution of (15a) *under $B_t = 0$.*

(Proof) See Appendix E.

Faced with production risks $\sigma > 0$, entrepreneurs produce less productive assets (capital) than in the economy without uninsured risks $\sigma = 0$. The growth rate decreases ($g_L < g_{NR}$). Uninsured capital production risks result in an inefficiently low growth rate. Reduced capital production increases the capital price $v_t = V_t^{-1}$ ($V_L < V_{NR}$) and hence, decreases the return on capital holdings ($r_L < r_{NR}$). The lowered return on holding productive assets creates a basis for bubbles.

3.2 Bubbly Economy

By using (15a) and (15b), we prove the following proposition, which shows the existence of a bubbly steady state.

Proposition 4 *Suppose that $\sigma > 0$.*
(i) If $A \leq \mu + 2\rho + 2\{\rho(\mu + \rho)\}^{1/2}$, no bubbly steady-state equilibrium exists.
(ii) If

$$A > \mu + 2\rho + 2\{\rho(\mu + \rho)\}^{1/2}, \tag{17}$$

there exist σ_1 and σ_2, where $0 < \sigma_1 < \sigma_2 < 1/(\rho + \mu)^{\frac{1}{2}}$, such that

(a) if $\sigma \notin (\sigma_1, \sigma_2)$, no bubbly steady-state equilibrium exists;

(b) if $\sigma \in (\sigma_1, \sigma_2)$, a unique bubbly steady-state equilibrium exists where $I_t > 0$ holds and V_t, B_t, r_t, ψ_t, and g_t satisfy

$$V_t = 1 - \sigma(\rho + \mu)^{\frac{1}{2}} \equiv V^* \qquad (\in (0,\ V_{NR})), \tag{18a}$$

$$B_t = \frac{A\left[1 - \sigma(\rho + \mu)^{\frac{1}{2}}\right]}{\frac{1}{\sigma}(\rho + \mu)^{\frac{1}{2}} - \mu} - 1 \equiv B^* \quad (> 0), \tag{18b}$$

$$r_t = AV^* - \delta \equiv r^*, \tag{18c}$$

$$\psi_t - r_t = \mu(1 + B^*) > 0, \tag{18d}$$

$$g_t = \frac{1 - V^*}{\sigma^2}(1 + B^*) - \delta \equiv g^*. \tag{18e}$$

(Proof) See Appendix F.

If the degree of entrepreneurial risks is in the middle range, $\sigma \in (\sigma_1, \sigma_2)$, asset bubbles exist in economies with advanced technology (A is large enough to satisfy (17)) (see Proposition 4 (ii)). If A is small, capital accumulates at a considerably low rate, which cannot sustain the expansion of asset bubbles.[14] With a large risk ($\sigma > \sigma_1$), entrepreneurs reduce the productive asset (capital) production considerably. The productive asset price increases and then, the rate of return on holding the productive asset r_t decreases, which leads to a positive risk premium on bubbly assets $\psi_t - r_t > 0$. Thus, entrepreneurs have an incentive to hold bubbly assets. Only if production risk is not too large ($\sigma < \sigma_2$), capital accumulates at a sufficiently high rate, and then, can sustain the expansion of asset bubbles. Thus, only for medium production risks ($\sigma \in (\sigma_1, \sigma_2)$), a bubbly steady state exists.

Our mechanism behind the existence of bubbles is different from that of existing models. We refer to infinitely lived agents models. In Kunieda and Shibata (2016), Hirano and Yanagawa (2017), and Miao and Wang (2018), occasionally binding borrowing constraints play a crucial role, which is absent from our model. In Aoki *et al.* (2014), where the return on *holding* productive assets (capital) incurs risks (see footnote 11), the rate of return on the bubbly assets is lower than that of capital. Entrepreneurs hold bubbly assets to diversify the risks of holding capital. This mechanism is also absent in our model because holding productive assets (capital) bears no risks and bubbly assets yield a higher return, $\psi_t > r_t$.

As in existing studies on rational bubbles, we can rewrite the existence condition of asset bubbles by using the growth rate and the rate of return on holding capital.

Proposition 5 *Suppose that $\sigma > 0$ and that a bubbleless steady-state equilibrium exists. A bubbly steady-state equilibrium exists if and only if $r_L < g_L - \mu$ holds.*

(Proof) See Appendix G.

Asset bubbles emerge if and only if r_L is sufficiently low to satisfy $r_L < g_L - \mu$ in the bubbleless steady state. Previous studies find similar conditions. Our mechanism behind the low rate of return on capital is different from that of previous studies. In OLG models, over-accumulation of capital results in a low interest rate. In the infinitely lived agents models of Kunieda and Shibata (2016), Hirano and Yanagawa (2017), and Miao and Wang (2018), borrowing constraints depress demand for borrowing, which lowers the interest rate. In Aoki *et al.* (2014), the risk premium on holding capital generates a low risk-free rate. In our model, the uninsured risks depress capital production and lower the rate of return on capital.

Existence of bubbles and comparative statics: To examine when bubbles are likely to exist, we prove the following proposition.

Proposition 6 *Suppose that there exist σ_1 and σ_2 given by Proposition 4. Then, we have*

$$(i)\ \ \frac{\partial \sigma_1}{\partial A} < 0, \quad \frac{\partial \sigma_2}{\partial A} > 0, \quad (ii)\ \ \frac{\partial \sigma_1}{\partial \rho} > 0, \quad \frac{\partial \sigma_2}{\partial \rho} < 0, \quad (iii)\ \ \frac{\partial \sigma_1}{\partial \mu} > 0, \quad \frac{\partial \sigma_2}{\partial \mu} < 0.$$

[14] We have $Y_t = AK_t \geq C_t \geq \rho p_t M$ because of (1), (10a), (10b), (12), $I_t \geq 0$, and $b_t^n = M$. Thus, $p_t M$ cannot grow faster than K_t ($\psi (\equiv \dot{p}_t / p_t) < g^*$). In the steady-state equilibrium, $\psi \leq g^*$ must hold.

(Proof) See Appendix H.

Condition (17) holds if A is large and if both ρ and μ are small. Hence, Propositions 4 and 6 indicate that asset bubbles are likely to arise in an economy in which technology is high, entrepreneurs are patient, and bubbles last long on average. The intuition is as follows. If we substitute (18c), (18d), and (18e) into (15b), we obtain $\dot{B}_t = (\psi^* - g^*)\, B_t$. Because $\dot{B}_t = 0$ in the bubbly steady state, the rate of return on bubbly assets is equal to the growth rate of the economy, $\psi^* = g^*$. With large A and small ρ, the economy grows rapidly. The rate of return on bubbly assets increases, which stimulates the speculative demand for bubbly assets. A small μ means a low probability of asset bubbles bursting. Even if the risk premium of holding bubbles is small, asset bubbles can exist.

3.3 Coexistence of Bubbly and Bubbleless Steady States

Because (17) implies (4), we immediately obtain the following corollary.

Corollary 1 *Suppose that $\sigma > 0$ and (17) holds. If $\sigma \in (\sigma_1, \sigma_2)$, two steady-state equilibria exist: the bubbly and bubbleless steady-state equilibria.*

The bubbly and bubbleless steady states coexist under the same parameter set. Figure 1 shows the phase diagram (see Appendix I). The bubbly steady state is unstable, while the bubbleless one is totally stable. The remainder of this section compares the properties of the bubbly steady state with those of the bubbleless steady state.

[Figure 1]

3.4 Growth Effects of Bubbles

The following proposition shows that even though there are no borrowing constraints and heterogeneity in productivity, asset bubbles may enhance long-run growth in our model.

Proposition 7 *Suppose that both the bubbly and bubbleless steady-state equilibria exist.*
(i) If $\mu + 2\rho + 2\{\rho(\mu + \rho)\}^{1/2} < A \leq 2(\mu + 2\rho)$, we have $g^* > g_L$.
(ii) If $A > 2(\mu + 2\rho)$, then there exists $\overline{\sigma} \in (\sigma_1, \sigma_2)$ such that
(a) if $\sigma = \overline{\sigma}$, we have $g^* = g_L$;
(b) if $\sigma \in (\sigma_1, \overline{\sigma})$, we have $g^* < g_L$;
(c) if $\sigma \in (\overline{\sigma}, \sigma_2)$, we have $g^* > g_L$.

(Proof) See Appendix J.

The growth effect of bubbles depends on the degree of risks, σ, and technology level, A. With relatively low technology (small A), bubbles always enhance long-run growth (Proposition 7 (i)). With advanced technology (large A), bubbles boost long-run growth if entrepreneurs face large risks (Proposition 7 (ii) (c)).

To interpret Proposition 7 intuitively, we rewrite the first-order condition for $I_{i,t}$, (B.14), as

$$(v_t - 1)I_{i,t} = \frac{(\sigma \cdot v_t I_{i,t})^2}{v_t k_{i,t} + b_{i,t}}. \tag{19}$$

The LHS shows the average profits from capital production. This represents entrepreneurs' incentive to increase $I_{i,t}$. The RHS shows the production risks $(\sigma v_t I_{i,t})^2$ relative to entrepreneur i's wealth, which represents the entrepreneurs' incentive to reduce $I_{i,t}$ to avoid production risks.

The asset bubble $b_{i,t}$ has a direct effect on the RHS. If v_t is constant, $b_{i,t}$ increases entrepreneur i's wealth from $v_t k_{i,t}$ to $v_t k_{i,t} + b_{i,t}$, which negatively affects the RHS. Asset bubbles make entrepreneurs wealthy, which encourages them to perform large-scale production in spite of large production risks.

In addition, asset bubbles indirectly affect (19) through v_t. The general good market equilibrium $Y_t = C_t + I_t$, or equivalently, (15a), shows this effect. The LHS of (15a) represents the general good supply, $Y_t = AK_t$. The RHS of (15a) shows that the general good demand $C_t + I_t$ decreases with the price of the general good relative to capital, $1/v_t (\equiv V_t)$ (see (10b) and (10c)). Asset bubbles make entrepreneurs wealthy and then increase the general good demand $C_t + I_t$. This raises the general good price relative to capital, $1/v_t$. Thus, we have $v_t^* < v_{L,t}$ (see Appendix G for the formal proof). Then, $(\sigma v_t I_{i,t})^2$ on the RHS of (19) decreases, which stimulates capital production. In addition, the decrease in v_t also reduces the LHS of (19), which depresses capital production.

Proposition 7 (i) and (ii)(c) hold as follows. A small A implies a low rental rate of capital (see (2)) and low capital price (in the bubbleless economy). $v_t k_{i,t}$ on the RHS of (19) becomes small, which decreases entrepreneurs' wealth in the bubbleless economy. Thus, the wealth effect of bubbles has a large impact on the RHS. If σ is large, the RHS of (19) becomes important, which also indicates a large wealth effect. Thus, in both cases, the positive growth effects dominate the negative one.

In infinitely lived agent models without borrowing constraints, Aoki *et al.* (2014) show that asset bubbles always decrease growth. Recent studies show that in the presence of borrowing constraints and heterogeneity in productivity, bubbles may increase growth. See Kocherlakota (2009), Kunieda and Shibata (2016), Hirano and Yanagawa (2017), and Miao and Wang (2018). These studies show that bubbles ease credit constraint of high-productive agents and thus promote growth. In our model, this mechanism is absent. Our model provides a different mechanism through which bubbles may enhance growth.

3.5 Amplification

The following proposition shows that bubbles amplify the effect of a marginal change in A.

Proposition 8 *Suppose that both bubbly and bubbleless steady-state equilibria exist. If $g^* \geq g_L$ holds, we have*

$$\frac{\partial g^*}{\partial A} > \frac{\partial g_L}{\partial A} > 0. \tag{20}$$

(Proof) See Appendix K.

(20) shows that a marginal technology shock has a larger impact in the bubbly economy than in the bubbleless economy. The result indicates that bubbles amplify the effect of A. The intuition is straightforward. With large A, capital grows rapidly. This is the direct effect of A. In the presence of bubbles, an increase in A has an indirect effect. The rapid capital

growth increases the rate of return on bubbly assets,[15] which stimulates the demand for bubbles. Then, the size of the bubbles expands. In fact, B^* increases with A (see (18b)). A large B^*, providing a large wealth effect, mitigates the production risks considerably, which indirectly amplifies the direct effect of A (if $g^* > g_L$ holds).

3.6 Collapse of Bubbles and Long-run Growth

We examine the effects of bubble crashes caused by a sunspot shock and a technology shock.

Sunspot shock: Suppose that Corollary 1 holds and the bubbly steady state prevails at time 0. At time $t_1 (> 0)$, a sunspot shock hits the economy. The shock follows a Poisson process with an arrival rate of μ. Then, asset bubbles burst. Since both V_t and B_t are jump variables, the economy immediately jumps to the bubbleless steady state.

The effect of the collapse of bubbles depends on σ and A. If either Proposition 7 (i) or (ii)(c) holds, then long-run growth decreases suddenly and permanently at time t_1. If Proposition 7 (ii)(b) holds, the collapse of bubbles increases growth at time t_1. Even without any fundamental changes, bubble crashes cause a sudden and permanent change in growth.

Technology shock: A large negative shock on A also collapses bubbles. Suppose that condition (17) holds and that the bubbly steady state prevails at time 0. At time $t_1 (> 0)$, A decreases unexpectedly and then violates condition (17). The bubbly steady state no longer exists. If condition (4) still holds, the economy jumps to the bubbleless steady state.

The decrease in A affects long-run growth through two channels. First, the decrease in A reduces growth directly. Second, the decreased growth cannot sustain bubbles and triggers the burst of asset bubbles. If Proposition 7 (i) or (ii)(c) holds, the collapse of bubbles depresses growth, which amplifies the first direct effect.

Moreover, even if the reduction in A is temporal, it may have a persistent effect. Suppose that A unexpectedly returns to its original level at time $t_2 (> t_1)$ and that after t_2, the bubbleless steady state continues to prevail. At time t_2, long-run growth recovers slightly. However, since bubbles no longer exist, long-run growth remains lower than in the initial economy. In summary, in the presence of bubbles, even a temporally negative technology shock has a substantial and persistent negative impact on long-run growth.

3.7 Bubbles and Welfare

We now investigate how bubbles affect the welfare of entrepreneurs. Initial aggregate capital is K_0. We have $\int k_{i,0} di = K_0$. Appendix L shows that in the bubbleless steady state, the utility of entrepreneur i at time 0 is

$$\rho W_L(k_{i,0}) = \log k_{i,0} + Z(g_L) - \frac{\sigma^2}{2\rho}(g_L + \delta)^2, \tag{21}$$

where $Z(g) \equiv \log\{A - (g + \delta)\} + g/\rho$. The first and second terms in $Z(g)$ represent the utility from today's consumption and consumption growth, respectively. Since entrepreneurs

[15] This can be observed as follows. From (15b), (18c), (18d), and (18e), we obtain $\psi^* = g^*$ in the bubbly steady state.

are risk averse, they suffer utility loss from entrepreneurial risks. This is reflected by the term $-\frac{\sigma^2}{2\rho}(g_L+\delta)^2$. Naturally, a large σ implies large utility loss.

Similarly, in the bubbly steady state, the utility of entrepreneur i at time 0 is given by

$$\rho W^*(k_{i,0}) = \log k_{i,0} + Z(g^*) - \frac{\sigma^2}{2\rho}\left(\frac{g^*+\delta}{1+B^*}\right)^2 - \mu\left[W^*(k_{i,0}) - W_L(k_{i,0})\right]. \tag{22}$$

The last term represents the utility loss of the bubble burst caused by a sunspot shock. The term $-\frac{\sigma^2}{2\rho}\left(\frac{g^*+\delta}{1+B^*}\right)^2$ captures the utility loss from capital creation risks. The term shows that B^* mitigates the utility loss from capital creation risks through the wealth effect.

From (21) and (22), we obtain

$$(\rho+\mu)\left[W^*(k_{i,0}) - W_L(k_{i,0})\right] = Z(g^*) - Z(g_L) + \frac{\sigma^2}{2\rho}\left\{(g_L+\delta)^2 - \left(\frac{g^*+\delta}{1+B^*}\right)^2\right\}. \tag{23}$$

The term $(g_L+\delta)^2 - \left(\frac{g^*+\delta}{1+B^*}\right)^2$ is always positive (see Appendix M), because bubbles mitigate capital creation risks and then have a positive welfare effect.

In Appendix M, we show that

$$Z(g^*) - Z(g_L) > (=)(<)0 \quad \Leftrightarrow \quad g^* - g_L > (=)(<)0. \tag{24}$$

As Proposition 3 shows, the production risks result in an inefficiently low growth rate. If bubbles enhance growth ($g^* > g_L$), then bubbles improve the efficiency of allocation between C_t and I_t, which has a positive welfare effect. In this case, the overall welfare effect is positive, $W^*(k_{i,0}) > W_L(k_{i,0})$.

If bubbles depress growth ($g^* < g_L$), then bubbles have a negative welfare effect. Even in this case, the positive effect of bubbles dominates the negative effect (see Appendix M). In summary, asset bubbles always improve the welfare of all entrepreneurs, mainly because bubbles mitigate the capital creation risks. We obtain the following proposition.

Proposition 9 *Suppose that both bubbly and bubbleless steady-state equilibria exist. Then, asset bubbles always improve the welfare of all entrepreneurs.*

(Proof) See Appendix M.

4 Discussion and Conclusion

We construct an infinitely lived agents model of rational bubbles. Our model is quite simple. To focus on uninsured risks, we assume that entrepreneurs receive the realization of production shocks only after they make decisions about production and portfolio choice. Even without occasionally binding borrowing constraints, asset bubbles emerge, depending on the degree of uninsured risks. Even without heterogeneity in productivity among entrepreneurs, bubbles enhance growth under some conditions. Moreover, the presence of bubbles amplifies a technology shock.

We briefly discuss the limitations of the present study and possible future works. First, many credit booms end with an economic crisis. The present study does not address how asset bubbles are related to credit booms. Thus, incorporating credit frictions into our model would be an important extension. Second, no policy interventions are considered in this study. It is said that contractionary monetary policy in Japan might have triggered the asset bubble burst around 1990. It would be interesting to investigate how policy interventions affect the existence of bubbles and the impacts of bubbles. Third, asset bubbles might be internationally contagious. Future works should consider the effects of bubbles in multi-country settings. Finally, the present study is purely qualitative. Thus, it is important to quantitatively examine the impact of the bubble burst.

Many authors have already tackled these issues, mainly by using OLG models.[16] However, since our infinitely lived agents model is quite simple and fairly close to the standard macroeconomic models that are widely used in modern macroeconomic literature, we consider our model to be a useful basis for these extensions and to provide new insights.

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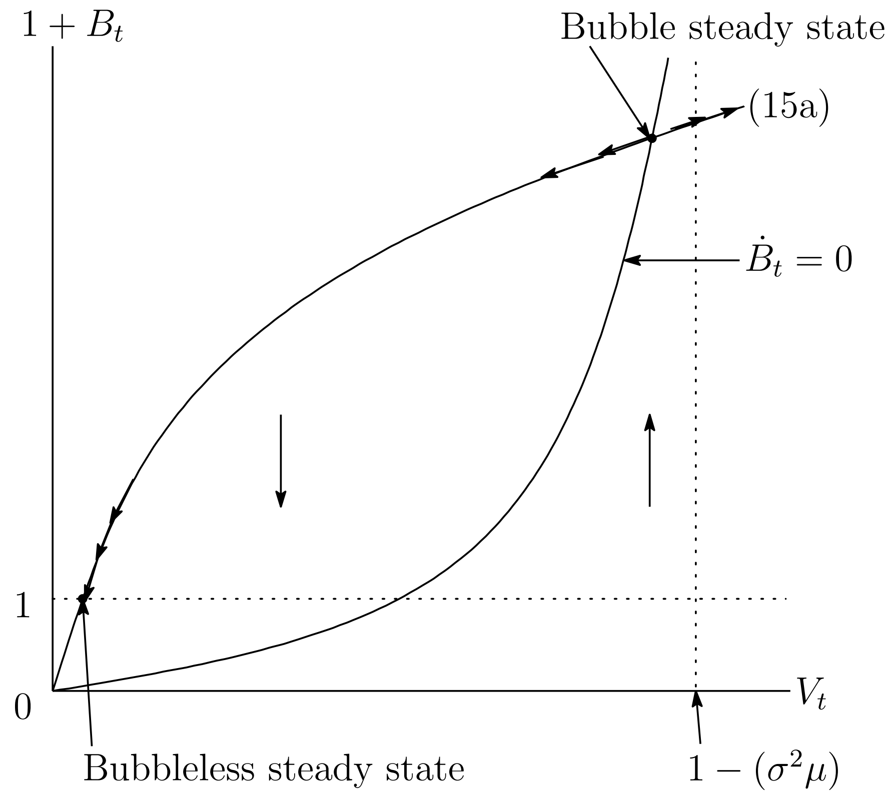

Figure 1 Phase Diagram

Appendix to "Asset Bubbles, Entrepreneurial Risks, and Economic Growth"

Takeo Hori[1] and Ryonghun Im.[2]

Appendix

A Derivation of the Budget Constraint (8)

Suppose that the bubbly economy prevails between t and $t+dt$. Between t and $t+dt$, entrepreneur i earns capital rental income $q_t k_{i,t} dt$ and profits given by (7). He or she consumes $c_{i,t}dt$ units of the general good, incurs capital depreciation $\delta \cdot v_t k_{i,t} dt$ ($\delta > 0$), and purchases $dk_{i,t}$ units of capital and $db^n_{i,t}$ units of bubbly assets. If he or she sells capital (bubbly assets), $dk_{i,t}$ ($db^n_{i,t}$) is negative. Thus, we have

$$c_{i,t}dt + \delta v_t k_{i,t} dt + v_t dk_{i,t} + p_t db^n_{i,t} = q_t k_{i,t} dt + (v_t - 1) I_{i,t} dt + \sigma v_t I_{i,t} dW_{i,t}. \tag{A.1}$$

From (5), we have $d\omega_{i,t} = (dv_t)k_{i,t} + v_t dk_{i,t} + (dp_t)b^n_{i,t} + p_t db^n_{i,t}$. By using $b_{i,t} = s_{i,t}\omega_{i,t}$, $a_{i,t} = (1 - s_{i,t})\omega_{i,t}$, (5), and (A.1), the budget constraint (8) is derived.

B Bellman Equation and the Optimal Behavior of an Entrepreneur

The discussion below (9d) shows that non-negativity constrains, $k_{i,t} \geq 0$ and $b^n_{i,t} \geq 0$, are always satisfied. Thus, this appendix ignores them. In the bubbly economy, let us denote the value function of entrepreneur i with $\omega_{i,t}$ by $U^*(\omega_{i,t}, t)$. In the bubbleless economy, we have $\omega_{i,t} = a_{i,t}$. Then, $U(a_{i,t}, t)$ is the value function for the bubbleless economy.

Following Stokey (2009, chapter 3), we derive the Bellman equations for $U^*(\omega_{i,t}, t)$ and $U(a_{i,t}, t)$. Consider an infinitesimally short time interval of length dt. Since bubbles burst with probability μdt, the Bellman equation of an entrepreneur with asset $\omega_{i,t}$ satisfies

$$U^*(\omega_{i,t}, t) = \max_{c_{i,t}, I_{i,t}, s_{i,t}} \Big\{ (\log c_{i,t}) dt + \frac{1}{1+\rho dt} E_t \left[(1 - \mu dt) \cdot U^*(\omega_{i,t+dt}, t+dt) + \mu dt \cdot U(a_{i,t+dt}, d+dt) \right] \Big\},$$

where $s_{i,t} = b_{i,t}/\omega_{i,t}$ and the maximization is subject to (5) and (8). We rearrange the above equation by using $U^*(\omega_{i,t+dt}, t+dt) = U^*(\omega_{i,t}, t) + dU^*(\omega_{i,t}, t)$ as follows:

$$\rho U^*(\omega_{i,t}, t) = \max_{c_{i,t}, I_{i,t}, s_{i,t}} \Big\{ (1 + \rho dt) \log c_{i,t} + E_t \left[\frac{dU^*(\omega_{i,t}, t)}{dt} - \mu (U^*(\omega_{i,t+dt}, t+dt) - U(a_{i,t+dt}, d+dt)) \right] \Big\}.$$

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Taking a limit of $dt \to 0$ in the above equation yields

$$\rho U^*(\omega_{i,t},t) = \max_{c_{i,t},I_{i,t},s_{i,t}} \left\{ \ln c_{i,t} + \frac{EdU^*(\omega_{i,t},t)}{dt} - \mu[U^*(\omega_{i,t},t) - U(a_{i,t},t)] \quad s.t \quad (5) \text{ and } (8) \right\}. \tag{B.1}$$

Similarly, in the bubbleless economy, we have

$$\rho U(a_{i,t},t) = \max_{c_{i,t},I_{i,t}} \left\{ \ln c_{i,t} + \frac{EdU(a_{i,t},t)}{dt} \quad s.t \quad (5) \text{ and } (8) \right\}. \tag{B.2}$$

We guess that $U^*(\omega_{i,t},t) = D^*(\ln \omega_{i,t} + u_t^*)$ and $U(a_{i,t},t) = D(\ln a_{i,t} + u_t)$. The asset holdings $\omega_{i,t}$ follow a stochastic process with a Brownian motion $W_{i,t}$. If we use Ito's lemma, the functional form of $dU^*(\omega_{i,t},t)$ is given by

$$dU^*(\omega_{i,t},t) = D^* \frac{d\omega_{i,t}}{\omega_{i,t}} - \frac{D^*}{2}\left(\frac{d\omega_{i,t}}{\omega_{i,t}}\right)^2 + D^* du_t^*. \tag{B.3}$$

From (8), $(d\omega_{i,t})^2$ is computed as follows:

$$\begin{aligned} (d\omega_{i,t})^2 &= \{[r_t(1-s_{i,t}) + \psi_t s_{i,t}]\,\omega_{i,t} + (v_t - 1)I_{i,t} - c_{i,t}\}^2 (dt)^2 \\ &\quad +2\{[r_t(1-s_{i,t}) + \psi_t s_{i,t}]\,\omega_{i,t} + (v_t - 1)I_{i,t} - c_{i,t}\}\,\sigma v_t I_{i,t} dt dW_{i,t} + (\sigma v_t I_{i,t} dW_{i,t})^2 \\ &= (\sigma v_t I_{i,t})^2 dt, \end{aligned} \tag{B.4}$$

where we use $(dt)^2 = 0$, $dtdW_{i,t} = 0$, and $(dW_{i,t})^2 = dt$. We substitute (8) and (B.4) into (B.3) and then take an expectation to obtain

$$\begin{aligned} E_t dU^*(\omega_{i,t},t) &= E_t \left\{ D^* \frac{\{[r_t(1-s_{i,t}) + \psi_t s_{i,t}]\,\omega_{i,t} + (v_t - 1)I_{i,t} - c_{i,t}\}\,dt + \sigma v_t I_{i,t} dW_{i,t}}{\omega_{i,t}} \right. \\ &\qquad \left. - \frac{D^*}{2}\left(\frac{\sigma v_t I_{i,t}}{\omega_{i,t}}\right)^2 dt + D^* du_t^* \right\} \\ &= D^* \frac{\{[r_t(1-s_{i,t}) + \psi_t s_{i,t}]\,\omega_{i,t} + (v_t - 1)I_{i,t} - c_{i,t}\}\,dt}{\omega_{i,t}} - \frac{D^*}{2}\left(\frac{\sigma v_t I_{i,t}}{\omega_{i,t}}\right)^2 dt + D^* du_t^*, \end{aligned} \tag{B.5}$$

where the second line uses $E_t dW_{i,t} = 0$. $E_t dU(a_{i,t},t)$ is given in the same manner.

The Bellman equation in the bubbleless economy is given by

$$\rho U(a_{i,t},t) = \max_{c_{i,t},I_{i,t}} \left\{ \log c_{i,t} + D\frac{r_t a_{i,t} + (v_t - 1)I_{i,t} - c_{i,t}}{a_{i,t}} - \frac{D}{2}\left(\frac{\sigma v_t I_{i,t}}{a_{i,t}}\right)^2 + D\dot{u}_t \right\}. \tag{B.6}$$

The first-order conditions are given by

$$c_{i,t} \ : \quad \frac{1}{c_{i,t}} = \frac{D}{a_{i,t}}, \tag{B.7}$$

$$I_{i,t} \ : \quad \frac{v_t - 1}{a_{i,t}} = \left(\frac{\sigma v_t}{a_{i,t}}\right)^2 I_{i,t}, \quad \sigma > 0. \tag{B.8}$$

If we use (B.6), (B.7), and (B.8), we obtain

$$\rho D \log a_{i,t} + \rho D u_t = \log a_{i,t} - \log D + D\left[r_t + \left(\frac{v_t - 1}{\sigma v_t}\right)^2\right] - 1 - \frac{D}{2}\left(\frac{v_t - 1}{\sigma v_t}\right)^2 + D\dot{u}_t. \quad (B.9)$$

Therefore, we obtain

$$D = \frac{1}{\rho}, \quad (B.10)$$

$$\rho u_t = \rho \ln \rho + r_t + \left(\frac{v_t - 1}{\sigma v_t}\right)^2 - \rho - \frac{1}{2}\left(\frac{v_t - 1}{\sigma v_t}\right)^2 + \dot{u}_t. \quad (B.11)$$

Then, we have

$$c_{i,t} = \rho a_{i,t}.$$

The transversality condition is satisfied:

$$\lim_{t\to\infty} E_t\left[\frac{a_{i,t}}{c_{i,t}} e^{-\rho t}\right] = \lim_{t\to\infty} \rho e^{-\rho t} = 0.$$

Next, we consider the Bellman equation in the bubbly economy. We distinguish the capital price in the bubbly economy v_t^* from that in the bubbleless economy v_t, because the existence of bubbly assets may affect the value of capital. If we use $U^*(\omega_{i,t}, t) = D^*(\log \omega_{i,t} + u_t^*)$ and $U(a_{i,t}, t) = D(\log a_{i,t} + u_t)$, then the Bellman equation in the bubbly economy can be written as

$$\begin{aligned} \rho U^*(\omega_{i,t}, t) = \max_{c_{i,t}, I_{i,t}, s_{i,t}} \Bigg\{ & \log c_{i,t} + D^* \frac{[r_t(1 - s_{i,t}) + \psi_t s_{i,t}]\,\omega_{i,t} + (v_t^* - 1)I_{i,t} - c_{i,t}}{\omega_{i,t}} \\ & - \frac{D^*}{2}\left(\frac{\sigma v_t^* I_{i,t}}{\omega_{i,t}}\right)^2 + D^* \dot{u}_t^* \\ & - \mu\left[D^*(\log \omega_{i,t} + u_t^*) - D\left(\log \frac{v_t}{v_t^*}(1 - s_{i,t})\omega_{i,t} + u_t\right)\right]\Bigg\}. \end{aligned} \quad (B.12)$$

The third line uses $a_{i,t} = v_t k_{i,t} = v_t(\omega_{i,t} - b_{i,t})/v_t^* = v_t(1 - s_{i,t})\omega_{i,t}/v_t^*$.

In the bubbly economy, the first-order conditions are given by

$$c_{i,t} : \quad \frac{1}{c_{i,t}} = \frac{D^*}{\omega_{i,t}}, \quad (B.13)$$

$$I_{i,t} : \quad \frac{v_t^* - 1}{\omega_{i,t}} = \left(\frac{\sigma v_t^*}{\omega_{i,t}}\right)^2 I_{i,t}, \quad (B.14)$$

$$s_{i,t} : \quad D^*(\psi_t - r_t) = D\frac{\mu}{1 - s_{i,t}}. \quad (B.15)$$

From (B.15), we obtain

$$s_{i,t} = 1 - \frac{D}{D^*}\frac{\mu}{\psi_t - r_t} = s_t. \quad (B.16)$$

Thus, all entrepreneurs hold the same fraction of their wealth as bubbly assets.

Using (B.13), (B.14), (B.15), and (B.16), we rewrite (B.12) as

$$\rho D^* \log \omega_{i,t} + \rho D^* u_t^* = \log \omega_{i,t} - \log D^* + D^* \left[r_t(1-s_t) + \psi_t s_t + \left(\frac{v_t^* - 1}{\sigma v_t^*} \right)^2 \right] - 1 + D^* \dot{u}_t^* - \frac{D^*}{2} \left(\frac{v_t^* - 1}{\sigma v_t^*} \right)^2 - \mu \left[D^*(\ln \omega_{i,t} + u_t^*) - D \left(\ln \frac{v_t}{v_t^*}(1-s_t)\omega_{i,t} + u_t \right) \right]. \tag{B.17}$$

Therefore, we obtain

$$D^* = \frac{1}{\rho}(= D), \tag{B.18}$$

$$\rho u_t^* = \rho \ln \rho + r_t(1-s_t) + \psi_t s_t + \left(\frac{v_t^* - 1}{\sigma v_t^*} \right)^2 - \rho - \frac{1}{2} \left(\frac{v_t^* - 1}{\sigma v_t^*} \right)^2 + \mu \left\{ \ln \left[(1-s_t) \frac{v_t^*}{v_t} \right] - u_t^* + u_t \right\} + \dot{u}_t^*. \tag{B.19}$$

The behavior of entrepreneur i is summarized by (9a)–(9b), and the transversality condition holds as $\lim_{t\to\infty} E_t \left[\frac{\omega_{i,t}}{c_{i,t}} e^{-\rho t} \right] = \lim_{t\to\infty} \frac{1}{\rho} e^{-\rho t} = 0$. Note that in these equations, we do not distinguish v_t^* from v_t for simplicity. Substituting (9a)–(9b) into (8) yields (9d).

C Proof of Proposition 1

Consider the case in which there are no risks concerning capital production $\sigma = 0$. From the first-order condition for $I_{i,t}$, (B.8) or (B.14), we obtain the first equation of (14). Hence, the capital price v_t is constant at $1(= \phi^{-1})$ and $\dot{v}_t = 0$. The rate of return on capital is given by the second equation of (14).

Next, we show that $B_t = 0$. If we use $B_t = p_t M/(v_t K_t)$, (10b), and the first equation of (14), the good market clearing condition (12) can be written as

$$A = \rho(1 + B_t) + \frac{I_t}{K_t}. \tag{C.1}$$

Because $I_t \geq 0$, $B_t = p_t M/(v_t K_t) \geq 0$ must be bounded above. Suppose that the price of bubbly assets is positive, $p_t > 0$. Then, we have

$$\begin{aligned} \dot{B}_t &= \left(\psi_t - \frac{\dot{K}_t}{K_t} \right) B_t \\ &= \{ \psi_t - A + \rho(1 + B_t) + \delta \} B_t \\ &= \{ \psi_t - r_t + \rho(1 + B_t) \} B_t \\ &= (\mu + \rho)(1 + B_t) B_t. \end{aligned} \tag{C.2}$$

The first line uses $v_t = 1$, $\dot{v}_t = 0$, and $\psi \equiv \dot{p}_t/p_t$. The second line uses (11) and (C.1). The third line uses $v_t = 1$, $dv_t = 0$, $r_t \equiv \frac{q + \dot{v}_t - \delta v_t}{v_t}$, and (2). The last line uses $v_t = 1$,

$a_{i,t} = (1 - s_t)\omega_{i,t}$, (9c), and

$$\frac{\mu}{\psi_t - r_t} = 1 - s_t = \frac{v_t K_t}{v_t K_t + p_t M} = \frac{1}{1 + B_t}. \tag{C.3}$$

Because $B_t \geq 0$ must be bounded, (C.2) is solved as $B_t = 0$. Thus, there is no bubbly equilibrium. From $B_t = 0$ and (C.1), we have $I_t/K_t = A - \rho > 0$ (the inequality holds because of (4)). From (11) and (C.1), we obtain the last equation of (14).

D Proof of Proposition 2

If $I_t > 0$, then (10c) holds. Substituting (5), (10b), and (10c) into (12), and after some rearrangement by using $B_t = p_t M/(v_t K_t)$, we obtain (15a).

In the bubbly economy $B_t > 0$, we can derive the dynamics of B_t as follows:

$$\frac{\dot{B}_t}{B_t} = \frac{\dot{p}_t}{p_t} - \frac{\dot{v}_t}{v_t} - \frac{\dot{K}_t}{K_t} = \mu(1 + B_t) + AV_t - \frac{1 - V_t}{\sigma^2}(1 + B_t).$$

In the second equality, we use (2), (10c), (11), and (C.3), $r_t \equiv \frac{q + \dot{v}_t - \delta v_t}{v_t}$, and $\psi_t \equiv \dot{p}_t/p_t$.

Note that in the bubbleless economy, we have $p_t = 0$, which implies that $B_t = \dot{B}_t = 0$. Then, (15b) holds in both the bubbly and bubbleless economies.

E Proof of Proposition 3

In the bubbleless economy, where $B_t = \dot{B}_t = 0$ holds, (15a) reduces to

$$A - \frac{\rho}{V_t} = \frac{1 - V_t}{\sigma^2}. \tag{E.1}$$

From (10c) and $V_t \equiv 1/v_t$, we know that in the bubbleless economy, we have $I_t/K_t = (1 - V_t)/\sigma^2$. Thus, if and only if the right-hand side (RHS) of (E.1) is positive, we have $I_t > 0$. In addition, we have $C_t = \rho K_t/V_t$ (see (10b)). Thus, $C_t > 0$ if and only if $V_t > 0$. We examine the condition under which (E.1) has a positive solution V_t ensuring that the RHS of (E.1) is positive.

The left-hand side (LHS) of (E.1) increases from zero to $A - \rho$ as V_t increases from ρ/A to 1 (see Figure A1). The RHS of (E.1) decreases from $1/\sigma^2$ to 0 as V_t increases from 0 to 1. Thus, if and only if $A - \rho > 0$, (E.1) has a unique solution $V_L \in \left(\frac{\rho}{A}, 1\right)$, ensuring that the RHS of (E.1) is positive, and hence, $I_t > 0$.

[Figure A1]

Substituting (2), $v_t = 1/V_L$, and $dv_t = 0$ into $r_t dt = (Adt - dv_t - \delta v_t dt)/v_t$ yields (16b). Because $V_L < 1 (\equiv V_{NR})$, we have $r_L < r_{NR}$. Substituting $\omega_t = v_t K_t = K_t/V_t$ and (10c) into (11) yields (16c). Because of (E.1), we can rewrite (16c) as

$$g_L = \left(A - \frac{\rho}{V_L}\right) - \delta. \tag{E.2}$$

Because $V_L < 1 (\equiv V_{NR})$, we have $g_L < g_{NR}$.

F Proof of Proposition 4

We first prove the following lemma.

Lemma A1 *Suppose that $\sigma > 0$. If and only if*

$$A\left[1-\sigma(\rho+\mu)^{\frac{1}{2}}\right] > \frac{1}{\sigma}(\rho+\mu)^{\frac{1}{2}} - \mu > 0, \tag{F.1}$$

there exists a unique bubbly steady-state equilibrium in which $I_t > 0$ holds and V_t, B_t, r_t, ψ_t, and g_t satisfy (18a), (18b), (18c), (18d), *and* (18e), *respectively.*

Proof: If we assume that $B_t > 0$, then (15b) and $\dot{B}_t = 0$ imply

$$AV_t = \left(\frac{1-V_t}{\sigma^2} - \mu\right)(1+B_t). \tag{F.2}$$

Solving (15a) and (F.2) for V_t yields $V = 1 \pm \sigma(\rho+\mu)^{1/2}$. Note that if we use (13), (10c) can be written as $I_t = (1-V_t)(1+B_t)K_t/\sigma^2$. To ensure $I_t > 0$, we must have $V_t < 1$. Thus, (18a) holds. From (C.3), we obtain (18d). Substituting (18a) into (F.2) yields (18b).

Condition (F.1) implies that $1 > \sigma(\rho+\mu)^{1/2}$, which ensures that $V^* > 0$. Condition (F.1) also ensures that $B^* > 0$. Thus, (F.1) ensures that $V^* > 0$ and $B^* > 0$.

Conversely, suppose that $V^* > 0$ and $B^* > 0$. Then, $V^* > 0$ implies that $1 > \sigma(\rho+\mu)^{1/2}$. Thus, $B^* > 0$ implies that condition (F.1).

Because V_t is constant at V^*, we obtain (18c) from $r = (A - \dot{v}_t - \delta)/v_t$. Substituting $\omega_t = v_t K_t + p_t M = (1+B_t)K_t/V_t$ and (10c) into (11) yields (18e). Lemma (A1) is proved. □

Note that (F.1) implies

$$\sigma < \min\left\{\frac{1}{(\rho+\mu)^{\frac{1}{2}}},\ \frac{(\rho+\mu)^{\frac{1}{2}}}{\mu}\right\}. \tag{F.3}$$

We also have

$$\frac{1}{(\rho+\mu)^{\frac{1}{2}}} < \frac{1}{(\rho+\mu)^{\frac{1}{2}}}\frac{\rho+\mu}{\mu} = \frac{(\rho+\mu)^{\frac{1}{2}}}{\mu}. \tag{F.4}$$

The inequality $\sigma < 1/(\rho+\mu)^{\frac{1}{2}}$ implies the second inequality in (F.1). Hence, (F.1) holds if and only if

$$\sigma < \frac{1}{(\rho+\mu)^{\frac{1}{2}}}, \tag{F.5}$$

$$A\left[1-\sigma(\rho+\mu)^{\frac{1}{2}}\right] > \frac{1}{\sigma}(\rho+\mu)^{\frac{1}{2}} - \mu. \tag{F.6}$$

Because $\sigma > 0$, we can rewrite (F.6) as

$$\Gamma(\sigma) \equiv A(\rho+\mu)^{\frac{1}{2}}\sigma^2 - (A+\mu)\sigma + (\rho+\mu)^{\frac{1}{2}} < 0. \tag{F.7}$$

Thus, the following lemma holds.

Lemma A2 *The bubbly steady-state equilibrium exists if and only if* (F.5) *and* (F.7) *hold.*

$\Gamma(\sigma)$ has the following properties:

$$\begin{aligned}\Gamma(0) &= (\rho+\mu)^{\frac{1}{2}} > 0,\\ \Gamma\left(\frac{1}{(\rho+\mu)^{\frac{1}{2}}}\right) &= \frac{\rho}{(\rho+\mu)^{\frac{1}{2}}} > 0,\\ \Gamma'(\sigma) &= 2A(\rho+\mu)^{\frac{1}{2}}\sigma - (A+\mu),\\ \Gamma'(0) &= -(A+\mu) < 0,\\ \Gamma'\left(\frac{1}{(\rho+\mu)^{\frac{1}{2}}}\right) &= A-\mu.\end{aligned} \tag{F.8}$$

Note that if $A-\mu \leq 0$, $\Gamma(\sigma)$ is a decreasing function for $\sigma \in (0,\ 1/(\rho+\mu)^{\frac{1}{2}})$. Because of (F.8), $\Gamma(\sigma) > 0$ holds for $\sigma \in (0,\ 1/(\rho+\mu)^{\frac{1}{2}})$ (see panel (a) of Figure A2). We obtain the following lemma.

Lemma A3 *Suppose that $A-\mu \leq 0$. Then, the bubbly steady-state equilibrium does not exist.*

[Figure A2]

Equation $\Gamma(\sigma) = 0$ has real solutions if and only if

$$\begin{aligned}0 &< (A+\mu)^2 - 4A(\rho+\mu)^{\frac{1}{2}}(\rho+\mu)^{\frac{1}{2}}\\ &= A^2 - 2(\mu+2\rho)A + \mu^2 \equiv H(A).\end{aligned} \tag{F.9}$$

Note the following points.

- If $H(A) \leq 0$, then $\Gamma(\sigma) \geq 0$ holds for all $\sigma > 0$ because of $\Gamma(0) > 0$. See panel (b) of Figure A2.
- If $H(A) > 0$ holds, then $\Gamma(\sigma) = 0$ has two solutions, σ_1 and σ_2. In addition, if $A > \mu$ holds, we have $\Gamma'(1/(\rho+\mu)^{\frac{1}{2}}) > 0$. Recall that $\Gamma(0) > 0$, $\Gamma(1/(\rho+\mu)^{\frac{1}{2}}) > 0$, and $\Gamma'(0) < 0$. Thus, we have $\sigma_1, \sigma_2 \in (0,\ 1/(\rho+\mu)^{\frac{1}{2}})$. In addition, $\Gamma(\sigma) < 0$ for $\sigma \in (\sigma_1, \sigma_2)$ and $\Gamma(\sigma) \geq 0$ for $\sigma \notin (\sigma_1, \sigma_2)$. See panel (c) of Figure A2.

From the discussion so far, we can prove the next lemma.

Lemma A4
(i) If $H(A) \leq 0$, there is no bubbly steady-state equilibrium.
(ii) If $A > \mu$ and $H(A) > 0$ hold, there are σ_1 and $\sigma_2 \in (0,\ 1/(\rho+\mu)^{\frac{1}{2}})$. If $\sigma \in (\sigma_1,\ \sigma_2)$, there exists a bubbly steady state. If $\sigma \notin (\sigma_1,\ \sigma_2)$, a bubbly steady state does not exist.

Next, we examine the properties of $H(A)$. We evaluate $H(A)$ at $A = 0$ and $A = \mu$ as follows:

$$H(0) = \mu^2 > 0, \tag{F.10}$$
$$H(\mu) = -4\rho\mu < 0. \tag{F.11}$$

Moreover, $H(A) = 0$ has the following solutions:

$$A = \mu + 2\rho + 2\{\rho(\mu + \rho)\}^{1/2} > \mu. \tag{F.12}$$

Figure A3 shows the graph of $H(A)$.

[Figure A3]

We obtain the following lemma.

Lemma A5
(i) If $A \leq \mu + 2\rho + 2\{\rho(\mu + \rho)\}^{1/2}$, then we have either $H(A) \leq 0$ or $A - \mu \leq 0$.
(ii) If $A > \mu + 2\rho + 2\{\rho(\mu + \rho)\}^{1/2}$, then we have both $H(A) > 0$ and $A - \mu > 0$.

From Lemmas A1–A5, we obtain Proposition 4.

G Proof of Proposition 5

We first show that $V_L < V^* < 1(\equiv V_{NR})$ holds. Suppose that both the bubbly and bubbleless steady states exist. From (15a), we have

$$A = \frac{\rho}{V_L} + \frac{1 - V_L}{\sigma^2} = \left[\frac{\rho}{V^*} + \frac{1 - V^*}{\sigma^2}\right](1 + B^*). \tag{G.1}$$

Since $B^* > 0$, the above relation implies that

$$\frac{\rho}{V_L} + \frac{1 - V_L}{\sigma^2} > \frac{\rho}{V^*} + \frac{1 - V^*}{\sigma^2}. \tag{G.2}$$

Because the LHS decreases with V_L, we have

$$V_L < V^*(\equiv 1 - \sigma(\rho + \mu)^{1/2}) < 1(\equiv V_{NR}). \tag{G.3}$$

Suppose that the stochastic bubbly steady state exists. Then, (18a) holds. We have

$$\begin{aligned}
(18a) \quad &\Longleftrightarrow \quad \left(\frac{1 - V^*}{\sigma}\right)^2 = \mu + \rho \\
&\Rightarrow \quad \left(\frac{1 - V_L}{\sigma}\right)^2 > \mu + \rho \\
&\Longleftrightarrow \quad r_L + \left(\frac{1 - V_L}{\sigma}\right)^2 - \rho > r_L + \mu \\
&\Longleftrightarrow \quad g_L > r_L + \mu.
\end{aligned} \tag{G.4}$$

The second line uses (G.3). In the bubbleless steady state, $\omega_t(= v_t K_t)$ grows at $g_L = \dot{K}_t/K_t$. If we integrate (9d) using the fact that $\omega_{i,t}$ and $dW_{i,t}$ are independent, we obtain the last line, because $s_t = 0$ holds in the bubbleless steady state.

Next, suppose that $g_L > r_L + \mu$ holds in the bubbleless steady state. From the second to the last lines of (G.4), we have $\left(\frac{1-V_L}{\sigma}\right)^2 > \mu + \rho$. Because $V_L < 1$, there exists $\hat{V}$ such that $\hat{V} > V_L$ and

$$\left(\frac{1-\hat{V}}{\sigma}\right)^2 = \mu + \rho \;\Rightarrow\; \hat{V} = 1 - \sigma(\mu+\rho)^{\frac{1}{2}} \equiv V^*. \tag{G.5}$$

Because $\hat{V} = V^* > V_L$, we obtain $B^* > 0$ from (G.1) and (G.2). Then, the bubbly steady state exists.

H Poof of Proposition 6

First, we consider the effects of ρ and A on σ_1 and σ_2. Recall that σ_1 and σ_2 are the solutions of $\Gamma(\sigma) = A(\rho+\mu)^{\frac{1}{2}}\sigma^2 - (A+\mu)\sigma + (\rho+\mu)^{\frac{1}{2}} = 0$ such that $\sigma_1 < \sigma_2$. The partial derivatives of $\Gamma(\sigma)$ with respect to ρ and A are given by

$$\frac{\partial \Gamma(\sigma)}{\partial \rho} = \frac{A\sigma^2 + 1}{2(\rho+\mu)^{\frac{1}{2}}} > 0, \tag{H.1}$$

$$\frac{\partial \Gamma(\sigma)}{\partial A} = (\rho+\mu)^{\frac{1}{2}}\sigma^2 - \sigma = \sigma\left[(\rho+\mu)^{\frac{1}{2}}\sigma - 1\right] < 0. \tag{H.2}$$

The inequality in (H.2) holds because of $1 - \sigma(\rho+\mu)^{\frac{1}{2}} > 0$ ($V^* > 0$). Because σ_1 and σ_2 are the solutions of $\Gamma(\sigma) = 0$, (H.1) and (H.2) imply that $\frac{\partial \sigma_1}{\partial \rho} > 0$, $\frac{\partial \sigma_2}{\partial \rho} < 0$ and $\frac{\partial \sigma_1}{\partial A} < 0$, $\frac{\partial \sigma_2}{\partial A} > 0$.

[Figure A4]

The partial derivative of $\Gamma(\sigma)$ with respect to μ is computed as follows:

$$\frac{\partial \Gamma(\sigma)}{\partial \mu} = \frac{A\sigma^2 + 1}{2(\rho+\mu)^{\frac{1}{2}}} - \sigma = \frac{A\sigma^2 - 2(\rho+\mu)^{\frac{1}{2}}\sigma + 1}{2(\rho+\mu)^{\frac{1}{2}}}. \tag{H.3}$$

We define $Q(\sigma) \equiv A\sigma^2 - 2(\rho+\mu)^{\frac{1}{2}}\sigma + 1$. Note that sign $\frac{\partial \Gamma(\sigma)}{\partial \mu}$ = sign $Q(\sigma)$ holds. Because of $A > \mu + 2\rho + 2\left[\rho(\rho+\mu)\right]^{\frac{1}{2}} > \mu + \rho$ (see (17)), $Q(\sigma)$ has a negative discriminant, $4\left[(\rho+\mu) - A\right] < 0$. Thus, $Q(\sigma) = 0$ does not have any real solutions. Because $Q(0) = \phi > 0$ holds, we have $Q(\sigma) > 0$ for all σ. Then, $\frac{\partial \Gamma(\sigma)}{\partial \mu} > 0$ holds. We obtain $\frac{\partial \sigma_1}{\partial \mu} > 0$, $\frac{\partial \sigma_2}{\partial \mu} < 0$.

I Phase Diagram

The resource constraint (15a) can be written as

$$1 + B_t = \frac{A}{\frac{\rho}{V_t} + \frac{1-V_t}{\sigma^2}} = \frac{AV_t}{\rho + \frac{(1-V_t)V_t}{\sigma^2}}.$$

The RHS increases from zero to A/ρ as V_t increases from 0 to 1. Since this equation represents the resource constraint, the economy is always on this line. We set $\dot{B}_t = 0$ in (15b) and then solve for $1 + B_t$ to obtain

$$1 + B_t = \frac{AV_t}{\frac{1-V_t}{\sigma^2} - \mu}.$$

The RHS increases from 0 to $+\infty$ as V_t increases from 0 to $1 - \mu\sigma^2$. In the region above (below) $\dot{B}_t = 0$ locus, we have $\dot{B}_t > 0$ ($\dot{B}_t < 0$). The phase diagram is shown in Figure 1 in Section 3.3. The phase diagram shows that the bubbly steady state is unstable, whereas the bubbleless one is stable.

J Proof of Proposition 7

We first show the following lemma.

Lemma A6 *If both the bubbly and bubbleless steady states exist, we have*

$$g^* < (=)(>) g_L \iff \sigma(\rho + \mu)^{1/2} < (=)(>) V_L.$$

(Proof) Irrespective of whether asset bubbles exist or not, (10c) and (15a) hold. We can rearrange (15a) as

$$\frac{1 + B_t}{V_t} = \frac{A}{\frac{(1-V_t)V_t}{\sigma^2} + \rho},$$

where $(V_t, B_t) = (V_L, 0)$ and $(V_t, B_t) = (V^*, B^*)$ hold in the bubbleless and bubbly economies, respectively. Then, (10c) can be written as

$$\frac{I_t}{K_t} = \frac{1 - V_t}{\sigma^2}(1 + B_t) = \frac{(1 - V_t)V_t}{\sigma^2} \frac{A}{\frac{(1-V_t)V_t}{\sigma^2} + \rho},$$

where $(V_t, B_t) = (V_L, 0)$ or $(V_t, B_t) = (V^*, B^*)$. Because $V_L \in (0,\ 1)$ and $V^* \in (0,\ 1)$, the above equation and (11) show that the growth rate increases with $(1 - V_t)V_t$. Thus, we have

$$\text{sign}\{g^* - g_L\} = \text{sign}\left\{(1 - V^*)V^* - (1 - V_L)V_L\right\}.$$

We have the following relationship:

$$\begin{aligned}
\text{sign}\{g^* - g_L\} &= \text{sign}\left\{(1 - V^*)V^* - (1 - V_L)V_L\right\} \\
&= \text{sign}\left\{\sigma(\rho + \mu)^{1/2}V^* - (V^* + \sigma(\rho + \mu)^{1/2} - V_L)V_L\right\} \\
&= \text{sign}\left\{\sigma(\rho + \mu)^{1/2}(V^* - V_L) - V_L(V^* - V_L)\right\} \\
&= \text{sign}\left\{[\sigma(\rho + \mu)^{1/2} - V_L](V^* - V_L)\right\} \\
&= \text{sign}\left\{\sigma(\rho + \mu)^{1/2} - V_L\right\}.
\end{aligned}$$

The second line uses $V^* \equiv 1 - \sigma(\rho + \mu)^{1/2}$. In the last line, we use $V^* > V_L$ (see (G.3)). Lemma A6 is proved. □

We next prove the following lemma.

Lemma A7

$$\sigma < (=)(>)\overline{\sigma} \iff V_L > (=)(<)\sigma(\rho+\mu)^{1/2},$$

where

$$\overline{\sigma} = \frac{-\mu + \sqrt{\mu^2 + 4A(\rho+\mu)}}{2A(\rho+\mu)^{1/2}} > 0. \tag{J.1}$$

(Proof) Note that V_L is a positive solution of (E.1). We evaluate both sides of (E.1) at $V_t = \sigma(\rho+\mu)^{1/2}$. As shown in Figure A5, we have the following relationship:

$$\begin{aligned} V_L < (=)(>)\sigma(\rho+\mu)^{1/2} &\iff A - \frac{\rho}{\sigma(\rho+\mu)^{1/2}} > (=)(<)\frac{1-\sigma(\rho+\mu)^{1/2}}{\sigma^2} \\ &\iff G(\sigma) \equiv A(\rho+\mu)^{1/2}\sigma^2 + \mu\sigma - (\rho+\mu)^{1/2} > (=)(<)0. \end{aligned}$$

[Figure A5]

Because $G(0) < 0$ and $G(\infty) > 0$, $G(\sigma) = 0$ has a unique positive solution $\overline{\sigma}$ that is defined by (J.1). Then, we have $\sigma < (=)(>)\overline{\sigma} \iff G(\sigma) < (=)(>)0 \iff V_L > (=)(<)\sigma(\rho+\mu)^{1/2}$. Lemma A7 is proved. □

The following lemma examines whether $\overline{\sigma} \in (\sigma_1,\ \sigma_2)$ holds.

Lemma A8

$$\overline{\sigma}\begin{cases} \notin (\sigma_1,\ \sigma_2), & \textit{if} \ \ \mu + 2\rho + 2\{\rho(\mu+\rho)\}^{1/2} < A \le 2(\mu+2\rho), \\ \in (\sigma_1,\ \sigma_2), & \textit{if} \ \ A > 2(\mu+2\rho). \end{cases}$$

(Proof) Recall that σ_1 and σ_2 are solutions of $\Gamma(\sigma) \equiv A(\rho+\mu)^{\frac{1}{2}}\sigma^2 - (A+\mu)\sigma + (\rho+\mu)^{\frac{1}{2}} = 0$ and that $\Gamma(\sigma) < 0$ holds if and only if $\sigma \in (\sigma_1,\ \sigma_2)$. Thus, the following relationship holds:

$$\overline{\sigma}\begin{cases} \in (\sigma_1,\ \sigma_2) & \text{if} \ \ \Gamma(\overline{\sigma}) < 0, \\ \notin (\sigma_1,\ \sigma_2) & \text{if} \ \ \Gamma(\overline{\sigma}) > 0. \end{cases} \tag{J.2}$$

We evaluate $\Gamma(\overline{\sigma})$ as follows:

$$\begin{aligned} \Gamma(\overline{\sigma}) &= A(\rho+\mu)^{\frac{1}{2}}\overline{\sigma}^2 - (A+\mu)\overline{\sigma} + (\rho+\mu)^{\frac{1}{2}} \\ &= A(\rho+\mu)^{\frac{1}{2}}\overline{\sigma}^2 - (A+\mu)\overline{\sigma} + A(\rho+\mu)^{1/2}\overline{\sigma}^2 + \mu\overline{\sigma} \\ &= \overline{\sigma}\{2A(\rho+\mu)^{\frac{1}{2}}\overline{\sigma} - A\} \\ &= \overline{\sigma}\left\{\sqrt{\mu^2 + 4A(\rho+\mu)} - (A+\mu)\right\}. \end{aligned}$$

The second line uses $G(\overline{\sigma}) \equiv A(\rho+\mu)^{1/2}\overline{\sigma}^2 + \mu\overline{\sigma} - (\rho+\mu)^{1/2} = 0$. The last line uses the definition of $\overline{\sigma}$, (J.1). Because of $\overline{\sigma} > 0$, we have

$$\begin{aligned} \Gamma(\overline{\sigma}) < (=)(>)0 &\iff \sqrt{\mu^2 + 4A(\rho+\mu)} < (=)(>)(A+\mu) \\ &\iff \mu^2 + 4A(\rho+\mu) < (=)(>)(A+\mu)^2 \\ &\iff A > (=)(<)2(\mu+2\rho). \end{aligned} \tag{J.3}$$

Recall that σ_1 and σ_2 exist if $A > \mu + 2\rho + 2\{\rho(\mu+\rho)\}^{1/2}$. Note that

$$\mu + 2\rho + 2[\rho(\mu+\rho)]^{1/2} < 2(\mu + 2\rho) \tag{J.4}$$

holds, because we have

$$\begin{aligned}
&2(\mu+2\rho) - \{\mu + 2\rho + 2[\rho(\mu+\rho)]^{1/2}\} = \mu + 2\rho - 2[\rho(\mu+\rho)]^{1/2}, \\
&(\mu+2\rho)^2 - \{2[\rho(\mu+\rho)]^{1/2}\}^2 = \mu^2 > 0.
\end{aligned}$$

From (J.2) and (J.3), $\overline{\sigma} \in (\sigma_1,\ \sigma_2)$ holds if $A > 2(\mu+2\rho)$, while $\overline{\sigma} \notin (\sigma_1,\ \sigma_2)$ holds if $\mu + 2\rho + 2\{\rho(\mu+\rho)\}^{1/2} < A \leq 2(\mu+2\rho)$. Lemma A8 is proved. □

Finally, we prove the next lemma.

Lemma A9 *If $\mu + 2\rho + 2\{\rho(\mu+\rho)\}^{1/2} < A \leq 2(\mu+2\rho)$, we have $\overline{\sigma} \leq \sigma_1$.*

(Proof) Lemma A8 implies that $\overline{\sigma} \notin (\sigma_1, \sigma_2)$. σ_1 and σ_2 are solutions of the quadratic equation $\Gamma(\sigma) = 0$ such that $\sigma_1 < \sigma_2$, where $\Gamma(\sigma)$ is defined by (F.7). The quadratic term σ^2 of $\Gamma(\sigma)$ has a positive coefficient. Thus, if $\overline{\sigma} \notin (\sigma_1, \sigma_2)$ satisfies $\Gamma'(\overline{\sigma}) < (>) 0$, we have $\overline{\sigma} \leq \sigma_1$ ($\overline{\sigma} \geq \sigma_2$). We evaluate $\Gamma'(\sigma)$ at $\sigma = \overline{\sigma}$.

$$\Gamma'(\overline{\sigma}) = \sqrt{\mu^2 + 4A(\rho+\mu)} - (A + 2\mu).$$

We define

$$\Psi(A) \equiv \left(\sqrt{\mu^2 + 4A(\rho+\mu)}\right)^2 - (A+2\mu)^2 = -A^2 + 4\rho A - 3\mu^2.$$

$\Gamma'(\overline{\sigma})$ and $\Psi(A)$ have the same signs. $\Psi(A)$ has the following properties:

1. The coefficient of A^2 in $\Psi(A)$ is negative.
2. $\Psi(\ \mu + 2\rho + 2\{\rho(\mu+\rho)\}^{1/2}\ ) = -4\{\mu^2 + \rho\mu + \mu[\rho(\rho+\mu)]^{1/2}\} < 0.$
3. $\Psi'(\ \mu + 2\rho + 2\{\rho(\mu+\rho)\}^{1/2}\ ) = -2\mu - 4\{\rho(\mu+\rho)\}^{1/2} < 0.$

Thus, we have $\Psi(A) < 0$ for $A \in (\mu + 2\rho + 2\{\rho(\mu+\rho)\}^{1/2}, 2(\mu+2\rho)]$ (see Figure A6). Then, $\Gamma'(\overline{\sigma}) < 0$ holds, which means that $\overline{\sigma} \leq \sigma_1$. Lemma A9 is proved. □

[Figure A6]

If $\mu + 2\rho + 2\{\rho(\mu+\rho)\}^{1/2} < A \leq 2(\mu+2\rho)$, Lemma A9 indicates that $\sigma > \overline{\sigma}$ holds for $\sigma \in (\sigma_1, \sigma_2)$. Lemmas A6 and A7 imply Proposition 7 (i).

If $A > 2(\mu+2\rho)$, Lemma A8 implies that $\overline{\sigma} \in (\sigma_1, \sigma_2)$. We obtain Proposition 7 (ii) from Lemmas A6 and A7.

K Proof of Proposition 8

From (18a), (18b), and (18e), we can easily show

$$\frac{\partial g^*}{\partial A} = \frac{g^* + \delta}{A} = \frac{1}{\sigma}(\rho+\mu)^{\frac{1}{2}}\frac{1+B^*}{A} > 0. \tag{K.1}$$

Recall that V_L is a positive solution of (E.1). Thus, we have $dA = -(\rho/V_L^2 + 1/\sigma^2)dV_L$. From (16c), we have $dg_L = -(1/\sigma^2)dV_L$. Thus, we have

$$\frac{\partial g_L}{\partial A} = \frac{1/\sigma^2}{\rho/V_L^2 + 1/\sigma^2} > 0.$$

We rearrange the above equation as follows:

$$\begin{aligned}
\frac{\partial g_L}{\partial A} &= \frac{\frac{1}{\sigma^2}(1-V^*)(1+B^*)}{(\rho/V_L^2 + 1/\sigma^2)(1-V^*)(1+B^*)} \\
&= \frac{g^*+\delta}{\frac{\rho}{V_L^2}(1-V^*)(1+B^*) + A - \frac{\rho(1+B^*)}{V^*}} \\
&= \frac{g^*+\delta}{A + \frac{\rho(1+B^*)}{V^* {V_L}^2}[V^*(1-V^*) - {V_L}^2]} \\
&= \frac{g^*+\delta}{A + \frac{\rho(1+B^*)}{V^* {V_L}^2}[V^* \cdot \sigma(\rho+\mu)^{\frac{1}{2}} - {V_L}^2]} \\
&< \frac{g^*+\delta}{A} = \frac{\partial g^*}{\partial A}.
\end{aligned}$$

The second line uses (15a) and (18e). The fourth line uses $V^* = \phi - \sigma(\rho+\mu)^{\frac{1}{2}}$. The inequality in the last line holds as follows. From (G.3), we have $V^* > V_L$. Lemma A6 implies that if $g^* \geq g_L$, then $\sigma(\rho+\mu)^{1/2} \geq V_L$. Thus, we have $V^* \cdot \sigma(\rho+\mu)^{\frac{1}{2}} - {V_L}^2 > 0$. Because $g^* + \delta > 0$ holds as shown in (K.1), the last inequality holds. Because $\partial g_L/\partial A > 0$, we have $\partial g^*/\partial A > \partial g_L/\partial A > 0$.

L Derivation of (21) and (22)

We first derive (21) and (22). In both the bubbly and bubbleless economies, we have $\omega_t = K_t/V_t + p_t M = (1+B_t)K_t/V_t$. Because both V_t and B_t are constant in the steady state, we have

$$g_t = \frac{\dot{K}_t}{K_t} = \frac{\dot{\omega}_t}{\omega_t} = r_t(1-s_t) + \psi_t s_t + \left(\frac{1-V_t}{\sigma}\right)^2 - \rho. \tag{L.1}$$

To obtain the last equality, we aggregate (9d) over i using the facts that $\omega_{i,t}$ and $dW_{i,t}$ are independent and $dW_{i,t}$ follows a normal distribution with mean zero.

Because $g_L = (1-V_L)/\sigma^2 - \delta$ holds in the bubbleless economy (see (16c)), we have

$$\frac{1-V_L}{\sigma} = \sigma(g_L + \delta). \tag{L.2}$$

From $s_t = 0$, (B.10), (B.11), (L.2), and $U(a_{i,t}, t) = D(\log a_{i,t} + u_t)$, we have

$$\rho U(a_{i,t}, t) = \log a_{i,t} + \log \rho + \frac{1}{\rho}\left[r_L + \left(\frac{1-V_L}{\sigma}\right)^2 - \rho\right] - \frac{1}{2\rho}\left\{\sigma(g_L+\delta)\right\}^2. \tag{L.3}$$

In the above equation, we use $\dot{u}_t = 0$, because r_t and v_t are constant. From (L.1) and (L.3), we obtain

$$\rho U(a_{i,0}, 0) = \log \rho a_{i,0} + \frac{1}{\rho}\left[g_L - \frac{1}{2}\left\{\sigma(g_L+\delta)\right\}^2\right]. \tag{L.4}$$

We have $\rho a_{i,0} = \rho v_0 k_{i,0} = (C_0/K_0)k_{i,0}$ in the bubbleless steady state. From (1), (11), and (12), we have

$$\frac{C_0}{K_0} = A - (g+\delta), \tag{L.5}$$

in both the bubbly and bubbleless economies. Thus, (L.4) is rewritten as (21).

Because $g^* = (1-V^*)(1+B^*)/\sigma^2 - \delta$ holds in the bubbly steady state, we have

$$\frac{1-V^*}{\sigma} = \frac{\sigma(g^*+\delta)}{1+B^*}. \tag{L.6}$$

From (B.18), (B.19), (L.6), and $U^*(\omega_{i,t}, t) = D^*(\log \omega_{i,t} + u_t^*)$, we have

$$\begin{aligned}\rho U^*(\omega_{i,t}, t) = \log \omega_{i,t} + \log \rho + \frac{1}{\rho}\left[r^*(1-s) + \psi s + \left(\frac{1-V^*}{\sigma}\right)^2 - \rho\right] \\ -\frac{1}{2\rho}\left\{\frac{\sigma(g^*+\delta)}{1+B^*}\right\}^2 - \mu\left[U^*(\omega_{i,t}, t) - U(a_{i,t}, t)\right].\end{aligned} \tag{L.7}$$

In the above equation, we use $\dot{u}_t^* = 0$, because r_t, v_t, s_t, and ψ_t are constant. From (L.1) and (L.7), we obtain

$$\rho U^*(\omega_{i,0}, 0) = \log \rho\omega_{i,0} + \frac{1}{\rho}\left[g^* - \frac{1}{2}\left\{\frac{\sigma(g^*+\delta)}{1+B^*}\right\}^2\right] - \mu\left[U^*(\omega_{i,0}, 0) - U(a_{i,0}, 0)\right]. \tag{L.8}$$

In the bubbly steady state, we have

$$\begin{aligned}\rho\omega_{i,0} &= \rho\omega_0\frac{\omega_{i,0}}{\omega_0} \\ &= C_0\frac{v_0 k_{i,0}}{v_0 K_0} \\ &= \frac{C_0}{K_0}k_{i,0}.\end{aligned} \tag{L.9}$$

The second equality uses $a_{0,i} = v_0 k_{i,0} = (1-s_0)\omega_{i,0}$, $K_0 = \int_0^1 k_{i,0}di$, and $\omega_0 = \int_0^1 \omega_{i,0}di$. From (L.5), (L.8), and (L.9), we obtain

$$(\rho+\mu)U^*(\omega_{i,0}, 0) = \log k_{i,0} + Z(g^*) - \frac{\sigma^2}{2\rho}\left(\frac{g^*+\delta}{1+B^*}\right)^2 + \mu W_L(k_{i,0}, 0) \equiv (\rho+\mu)W^*(k_{i,0}, 0).$$

After rearranging the above equation, (22) is derived.

M Proof of Proposition 9

From (L.2) and (L.6), we have

$$\text{sign}\ \left\{(g_L+\delta)^2-\left(\frac{g^*+\delta}{1+B^*}\right)^2\right\} = \text{sign}\ \left\{(1-V_L)^2-(1-V^*)^2\right\}. \tag{M.1}$$

From (G.3), we have $V_L < V^* < 1$. Thus, $(1-V_L)^2 > (1-V^*)^2$ holds.

We show that $\max\{g_L, g^*\} < g_{NR}$, where $g_{NR} \equiv A-\delta-\rho$ is the growth rate under $\sigma = 0$ (see the last equation of (14)). (16c) ensures that $g_L < g_{NR}$. Using (18e), we show $g^* < g_{NR}$ as follows:

$$\begin{aligned} g^* &= \left(A-\rho\frac{1+B^*}{V^*}\right)-\delta \\ &< \left(A-\rho\frac{1}{V^*}\right)-\delta \\ &< A-\rho-\delta \equiv g_{NR}. \end{aligned} \tag{M.2}$$

The first line uses (15a) in (18e). The second line uses $B^* > 0$. The last line uses $V^* < 1$.

Function $Z(g)$ has the following properties:

$$Z'(g) = \frac{g_{NR}-g}{\rho(g_{NR}+\rho-g)} > 0 \quad \text{for} \quad g < g_{NR}, \tag{M.3}$$

$$Z''(g) = \frac{-1}{(g_{NR}+\rho-g)^2} < 0 \quad \text{for} \quad g < g_{NR}. \tag{M.4}$$

We consider the following two cases: (i) $g^* \geq g_L$ and (ii) $g^* < g_L$.

(i) If $g^* \geq g_L$ holds, we have $Z(g^*) \geq Z(g_L)$ because of (M.2) and (M.3). Since $(1-V_L)^2 > (1-V^*)^2$ holds, we have $U^*(\omega_{i,0}, 0) > U(a_{i,0}, 0)$.

(ii) If $g^* < g_L$ holds, we have $Z(g^*) < Z(g_L)$ because of $\max\{g_L, g^*\} < g_{NR}$ and (M.3). Furthermore, since $Z(g)$ is an increasing and concave function for $g < g_{NR}$, we have

$$0 < Z(g_L) - Z(g^*) < Z'(g^*)(g_L - g^*). \tag{M.5}$$

In addition, because of $V_L < V^*$ (see (G.3)), we have

$$\begin{aligned} (1-V_L)^2-(1-V^*)^2 &= (1-V_L+1-V^*)(V^*-V_L) \\ &> 2(1-V^*)(V^*-V_L). \end{aligned} \tag{M.6}$$

From (23), (M.5), and (M.6), we obtain

$$(\rho+\mu)\left[U^*(\omega_{i,0},0)-U(a_{i,0},0)\right] > Z'(g^*)(g^*-g_L)+\frac{1}{\rho\sigma^2}(1-V^*)(V^*-V_L). \tag{M.7}$$

If we use $g_{NR} \equiv A-\delta-\rho$ and the first line of (M.2), we have

$$Z'(g^*) = \frac{1}{\rho}\left(1-\frac{\rho}{A-\delta-g^*}\right) = \frac{1}{\rho}\left(1-\frac{V^*}{1+B^*}\right). \tag{M.8}$$

Using (16c), (18e), and (M.8), we examine the sign of the second line of (M.7) as follows:

$$
\begin{aligned}
&\operatorname{sign}\left\{Z'(g^*)(g^*-g_L)+\frac{1}{\rho\sigma^2}(1-V^*)(V^*-V_L)\right\}\\
&=\operatorname{sign}\left\{\left(1-\frac{V^*}{1+B^*}\right)\{(1-V^*)(1+B^*)-(1-V_L)\}+(1-V^*)(V^*-V_L)\right\}\\
&=\operatorname{sign}\left\{(1-V^*)-V_L(1-V^*)+(1-V^*)B^*-(1-V_L)+\frac{V^*(1-V_L)}{1+B^*}\right\}\\
&=\operatorname{sign}\left\{(1-V_L)(1-V^*)+(1-V^*)B^*+\frac{(V^*-1)(1-V_L)-(1-V_L)B^*}{1+B^*}\right\}\\
&=\operatorname{sign}\left\{\frac{(1-V_L)(1-V^*)B^*}{1+B^*}+(1-V^*)B^*-\frac{(1-V_L)B^*}{1+B^*}\right\}\\
&=\operatorname{sign}\left\{(1-V^*)-\frac{(1-V_L)V^*}{1+B^*}\right\}\\
&=\operatorname{sign}\left\{\frac{(1-V^*)}{V^*}(1+B^*)-(1-V_L)\right\}\\
&=\operatorname{sign}\left\{A\frac{\sigma(\rho+\mu)^{\frac{1}{2}}}{\frac{1}{\sigma}(\rho+\mu)^{\frac{1}{2}}-\mu}-(1-V_L)\right\}. \qquad \text{(M.9)}
\end{aligned}
$$

The last line uses (18a) and (18b). Note that the first term in the last line of (M.9) increases with μ. Because $\mu > 0$, we have

$$
\begin{aligned}
A\frac{\sigma(\rho+\mu)^{\frac{1}{2}}}{\frac{1}{\sigma}(\rho+\mu)^{\frac{1}{2}}-\mu}-(1-V_L) &> \sigma^2\left(A-\frac{1-V_L}{\sigma^2}\right)\\
&=\frac{\rho\sigma^2}{V_L}\\
&>0. \qquad \text{(M.10)}
\end{aligned}
$$

The second line uses (E.1).

From (M.7), (M.9), and (M.10), we obtain $U^*(\omega_{i,0},0) > U(a_{i,0},0)$.

N A Variety Expansion Model

In our model, we can interpret capital broadly. To observe this fact, this appendix modifies the variety expansion model proposed by Barro and Sala-i-Martin (2004). Entrepreneurs can set up new businesses or develop new technologies, which are subject to idiosyncratic shocks. The number of firms in the economy accumulates through entrepreneurial activities. The main results obtained in our AK model hold in this variety-expansion model. Thus, in our model, capital, K, can include not only physical capital but also businesses and innovations.

A general good is produced by using intermediate goods and labor. Labor is supplied inelastically by workers. Entrepreneurs can create new firms and accumulate their own firms.

Production sector: A competitive general good firm has the following production technology:

$$Y_t = ZL_t^{\alpha} \int_0^{n_t} X_t^{1-\alpha}(j)dj, \quad Z > 0, \quad 0 < \alpha < 1, \tag{N.1}$$

where n_t is the number of varieties, and L_t and $X_t(j)$ represent labor and intermediate good j inputs, respectively. Profit maximization yields $X_t(j) = [(1-\alpha)Z]^{\frac{1}{\alpha}} L_t p_t^X(j)^{-\frac{1}{\alpha}}$, where $p_t^X(j)$ denotes the price of intermediate good j. See Appendix N.1 for the derivation of the optimal behavior of general and intermediate good firms.

Each intermediate good j is produced by a monopolistically competitive firm. The production of one unit of intermediate good requires $\eta > 0$ units of general goods. The profit of each intermediate good is given by $\pi_t(j) = \left[p_t^X(j) - \eta\right] X_t(j)$. Appendix N.1 shows that from the profit maximization problem by firm j, we obtain

$$\pi_t(j) = \alpha \left[\eta^{\alpha-1}(1-\alpha)^{2-\alpha} Z\right]^{\frac{1}{\alpha}} L \equiv \pi, \tag{N.2}$$

where we use the labor market condition $L_t = L$, where L is the total labor supply. Thus, π is constant over time.

Entrepreneurs: Entrepreneurs create new firms using their technology given by $dx_{i,t}^N = I_{i,t}dt + \sigma I_{i,t} dW_{i,t}$, where $dx_{i,t}^N$ denotes the number of newly created firms by entrepreneur i and includes starting new businesses and developing new technologies. Entrepreneurs are the owners of intermediate good firms. Entrepreneur i owns $n_{i,t}$ units of intermediate good firms. The market value of an intermediate good firm is v_t^N. Then, the total asset holdings of entrepreneurs i are given by $\omega_{i,t}^N = v_t^N n_{i,t} + p_t b_{i,t}^n = a_{i,t}^N + b_{i,t}$, where $a_{i,t}^N \equiv v_t^N n_{i,t}$. Between $t+dt$, entrepreneur i receives operating profits from intermediate good firms $\pi n_{i,t} dt$ and earns profit income by creating new intermediate good firms $v_t^N dx_{i,t}^N - I_{i,t}dt = (v_{i,t}^N - 1)I_{i,t}dt + \sigma v_t^N I_{i,t} dW_{i,t}$. The budget constraint of entrepreneur i at t is as follows:

$$d\omega_{i,t}^N = \left\{\left[r_t^N(1-s_{i,t}) + \psi_t s_{i,t}\right]\omega_{i,t} - c_{i,t}\right\} dt + (v_t^N - 1)I_{i,t}dt + \sigma v_t^N I_{i,t} dW_{i,t}, \tag{N.3}$$

where $r_t^N dt \equiv (\pi dt + dv_t^N - \delta v_t^N dt)/v_t^N$ and $\psi_t dt = dp_t/p_t$. $s_{i,t} \equiv b_{i,t}/\omega_{i,t}$ denotes the portfolio weight of bubbly assets. The detailed derivation of the budget constraint, the evolution of $\omega_{i,t}^N$, and the optimal plans of entrepreneur i are shown in Appendix N.2.

Workers: The population size of workers is L. Each worker inelastically supplies one unit of labor and earns wage income. For simplicity, we assume that the workers are hand-to-mouth consumers, which means that they consume their current income entirely. Then, the aggregate consumption of workers, C_t^w, is given by

$$C_t^w = w_t L, \tag{N.4}$$

where w_t denotes the wage rate.

Equilibrium: Let us define $V_t^N \equiv 1/v_t^N$ and $B_t^N \equiv p_t M/(v_t^N n_t)$, where B_t^N denotes the value of bubbles relative to the market value of intermediate goods firms, and $n_t \equiv \int_0^1 n_{i,t} di$ denotes the aggregate number of intermediate good firms. Then, the aggregate asset holdings are given by $\omega_t^N = v_t^N n_t + p_t M$, where we use $\int_0^1 b_{i,t}^n di = M$. We derive the law of motion of the number of firms as $dn_t \equiv \int_0^1 (dx_{i,t}^N)di - \delta n_t dt = (I_t - \delta n_t)dt$, where I_t is given by (N.19). The growth rate of the economy is as follows:

$$g_t^N = \frac{\dot{n_t}}{n_t} = \frac{I_t}{n_t} - \delta. \tag{N.5}$$

The following proposition gives a set of equations that characterize the bubbly and bubbleless equilibria in the variety expansion model.

Proposition A1 *Assume that $\sigma > 0$. Then, the bubbly and bubbleless equilibria with $I_t > 0$ are characterized by*

$$\pi = \left[\frac{\rho}{V_t^N} + \frac{1 - V_t^N}{\sigma^2}\right](1 + B_t^N), \tag{N.6}$$

$$\dot{B_t^N} = \left[\mu(1 + B_t^N) + \pi V_t^N - \frac{1 - V_t^N}{\sigma^2}(1 + B_t^N)\right] B_t^N, \tag{N.7}$$

where π is given by (N.2).

Proof: See Appendix N.3.

We compare (N.6) and (N.7) with (15a) and (15b), respectively. If we replace A with π in (15a) and (15b), these two equations become equivalent to (N.6) and (N.7), respectively. This means that if we substitute π into A, Propositions 3–7 and Corollary 1 hold even in this variety-expansion model. This fact shows that in our AK model, capital K includes not just physical capital but also businesses and innovations.

N.1 The Optimal Behavior of Production Sector

The profits of the general good firm are given by $\pi_t = Y_t - \int_0^{n_t} p_t^X X_t(j)dj - w_t L_t$. The first-order conditions are given by

$$X_t(j) \;:\; (1-\alpha)ZL_t^{\alpha}X_t^{-\alpha}(j) = p_t^X(j), \quad j \in [0, n_t], \tag{N.8}$$

$$L_t \;:\; \alpha Z L_t^{\alpha-1}\int_0^{n_t} X_t^{1-\alpha}(j)dj = w_t. \tag{N.9}$$

From (N.8), we obtain

$$X_t(j) = [(1-\alpha)Z]^{\frac{1}{\alpha}} L_t p_t^X(j)^{-\frac{1}{\alpha}}, \quad j \in [0, n_t]. \tag{N.10}$$

The profits of intermediate good firm j are $\pi_t(j) = \left[p_t^X(j) - \eta\right] X_t(j)$. Intermediate good firm j maximizes the profits subject to (N.10). We obtain the following equations:

$$p_t^X(j) = \frac{\eta}{1-\alpha} \equiv p^X, \tag{N.11}$$

$$X_t(j) = \left[\frac{(1-\alpha)^2 Z}{\eta}\right]^{\frac{1}{\alpha}} L \equiv X, \tag{N.12}$$

where $L_t = L$. From (N.11), (N.12), and $\pi_t(j) = \left[p_t^X(j) - \eta\right] X_t(j)$, we have (N.2).

N.2 The Evolution of $\omega_{i,t}^N$ and Optimal Behavior of an Entrepreneur in the Variety Expansion Model

First, we derive the budget constraint of entrepreneur i using the procedure presented in (8). As in (A.1), we have

$$c_{i,t}dt + \delta v_t^N n_{i,t}dt + v_t^N dn_{i,t} + p_t db_{i,t}^n = \pi n_{i,t}dt + (v_t^N - 1)I_{i,t}dt + \sigma v_t^N I_{i,t}dW_{i,t}, \qquad \text{(N.13)}$$

where $\delta \in [0,1]$ denotes the exogenous destruction rate of an intermediate good firm and π corresponds to the rental price q of the AK model presented in Section 2. From $\omega_{i,t}^N = v_t^N n_{i,t} + p_t b_{i,t}^n$, we obtain $d\omega_{i,t}^N = (dv_t^N)n_{i,t} + v_{i,t}^N(dn_{i,t}) + (dp_t)b_{i,t}^n + p_t(db_{i,t}^n)$. Then, (N.13) can be rearranged as

$$d\omega_{i,t}^N = \left[r_t^N a_{i,t}^N + \psi_t b_{i,t} - c_{i,t}\right]dt + (v_t^N - 1)I_{i,t} + \sigma v_t^N I_{i,t}dW_{i,t}, \qquad \text{(N.14)}$$

where $a_t^N \equiv v_t^N n_{i,t}$ and $r_t^N dt = (\pi dt + dv_t^N - \delta v_t^N dt)/v_t^N$, and we use $\psi dt = dp_t/p_t$. By employing $a_{i,t} = (1 - s_{i,t})\omega_{i,t}$, $b_{i,t} = s_{i,t}\omega_{i,t}$, and (N.14), we obtain (N.3).

Entrepreneur i maximizes (3) subject to $\omega_{i,t}^N = v_t^N n_{i,t} + p_t b_{i,t}^n$ and (N.3). If we replace $\omega_{i,t}$, r_t, and $v_{i,t}$ with $\omega_{i,t}^N$, r_t^N, and v_t^N in (8), (N.3) is equivalent to (8). By using the procedure presented in Appendix B, we can derive the optimal behavior as follows:

$$c_{i,t} = \rho\omega_{i,t}^N, \qquad \text{(N.15)}$$

$$s_{i,t} = s_t = \begin{cases} 1 - \frac{\mu}{\psi_t - r_t^N} & \text{in the bubbly economy } (p_t > 0), \\ 0 & \text{in the bubbleless economy } (p_t = 0), \end{cases} \qquad \text{(N.16)}$$

$$I_{i,t} = \frac{v_t^N - 1}{(\sigma v_t^N)^2}\omega_{i,t}^N, \quad \sigma > 0. \qquad \text{(N.17)}$$

If $v_t^N > 1$, then $I_{i,t} > 0$ holds. The transversality condition holds $\lim_{t\to\infty} E_t\left[\frac{\omega_{i,t}^N}{c_{i,t}}e^{-\rho t}\right] = \lim_{t\to\infty}\frac{1}{\rho}e^{-\rho t} = 0$.

N.3 Proof of Proposition A1

From (N.15), (N.17), $a_{i,t}^N = (1 - s_{i,t})\omega_{i,t}$, and $\omega_t^N = v_t^N n_t + p_t M$, the aggregate consumption and investment are given by

$$C_t = \rho\omega_t^N, \qquad \text{(N.18)}$$

$$I_t = \frac{v_t^N - 1}{(\sigma v_t^N)^2}\omega_t^N, \quad \sigma > 0, \qquad \text{(N.19)}$$

respectively. The market clearing condition for general goods is given by

$$Y_t = C_t + C_t^w + I_t + \eta X n_t, \qquad \text{(N.20)}$$

where we use $X_t(j) = X$ from (N.12). Because the general good sector is competitive, $Y_t = p^X X n_t + w_t L$ holds. Then, (N.20) can be rewritten as

$$\begin{aligned} & p^X X n_t + w_t L = C_t + C_t^w + I_t + \eta X n_t \\ \iff & (p^X - \eta)X n_t = C_t + I_t \\ \iff & \pi n_t = \rho\omega_t^N + \frac{v_t^N - 1}{(\sigma v_t^N)^2}\omega_t^N. \end{aligned} \qquad \text{(N.21)}$$

The first line uses $Y_t = p^X X n_t + w_t L$. The second line uses (N.4). The third line uses $\pi = (p^X - \eta)X$, (N.18), and (N.19). Dividing both sides of (N.21) by n_t and after some rearrangement by using $\omega_t^N = v_t^N n_t + p_t M$, $V_t^N = 1/v_t^N$ and $B_t^N = p_t M/(v_t^N n_t)$, (N.6) is derived.

The dynamics of asset bubbles are given by $\dot{B}_t^N / B_t^N = \dot{p}_t/p_t - \dot{v}_t^N/v_t^N - \dot{n}_t/n_t$, where $\dot{n}_t/n_t$ is given by (N.5). Using the procedure presented in the derivation of (15b), (N.7) is obtained. □

O Neoclassical Economy

This appendix shows that our main results obtained in the AK model are not affected even if we use neoclassical production technology and assume exogenous growth. Entrepreneurs face the same utility maximization problem as in the AK model. Their behavior is again given by (9a)-(9d) and thus, (10a)-(10c) hold again. As in Appendix N, we assume the existence of hand-to-mouth workers.

Production sector: Production technology of the general good is given by

$$Y_t = F(K_t, H_t L_t), \quad H_t > 0, \tag{O.1}$$

where H_t grows over time, and $g_H \equiv \frac{\dot{H}_t}{H_t}$ represents an exogenous growth rate of H_t. The production function F satisfies the standard neoclassical assumptions: F is continuous, with constant returns to scale, exhibiting positive and diminishing marginal products with respect to K_t and L_t, and satisfying the Inada conditions and $F(0, H_t L_t) = F(K_t, 0) = 0$. Under a competitive economy, K_t and L_t are paid their marginal products, $q_t = \partial F/\partial K_t$ and $w_t = H_t \partial F/\partial L_t$, respectively.

Condition for asset bubbles in the neoclassical economy: The equilibrium dynamics in both the bubbleless and bubbly economies are given in Appendix O.1. We show how the degree of insurance incompleteness σ affects the existence of asset bubbles in the neoclassical growth economy.

Proposition A2 *Suppose that $\sigma > 0$. Then the following holds.*
(i) If $\sigma \le \frac{(\rho+\mu)^{\frac{1}{2}}}{g_H+\delta}$, then only the bubbleless steady-state equilibrium exists.
(ii) If $\frac{(\rho+\mu)^{\frac{1}{2}}}{g_H+\delta} < \sigma < \frac{1}{(g_H+\delta)^{\frac{1}{2}}}$, then both bubbly and bubbleless steady-state equilibria exist.
(iii) If $\frac{1}{(g_H+\delta)^{\frac{1}{2}}} \le \sigma < \frac{1}{(\rho+\mu)^{\frac{1}{2}}}$, then only the bubbly steady-state equilibrium exists.
(iv) If $\frac{1}{(\mu+\rho)^{\frac{1}{2}}} \le \sigma$, then there is neither a bubbly nor a bubbleless steady-state equilibrium.

Proof: See Appendix O.2.

Proposition A2 (*ii*) and (*iii*) show that the bubbly steady state exists for medium degrees of risks σ. In addition, Proposition A2 (*ii*) shows that both steady states coexist. These results are consistent with Proposition 4 in the AK model and Corollary 1. The difference from Proposition 4 is the case of (*iii*). In this case, a bubbly steady state exists and no bubbleless steady state exists in the neoclassical economy.

Capital and consumption effects of bubbles: We examine the effects of asset bubbles on capital accumulation and consumption in both steady states. We focus on total consumption, $C_t^T \equiv C_t + C_t^w$, where C_t and C_t^w are given by (10b) and (N.4), respectively. Define $k_{H,t} \equiv K_t/H_t$ and $c_t^T \equiv C_t^T/H_t$. As in the AK model, an asterisk $*$ and subscript L represent the steady-state values in the bubbly and bubbleless steady-state equilibria, respectively. We show the following proposition.

Proposition A3 *Suppose that bubbly and bubbleless steady states exist, that is, the case of (ii) in Proposition A2 holds. Then, the following hold.*
(i) If $2(\mu+\rho) \geq g_H + \delta$, *then* $k_H^* > k_{H,L}$ *and* $c^{T*} > c_L^T$ *hold for* $\sigma \in (\frac{(\rho+\mu)^{\frac{1}{2}}}{g_H+\delta}, \frac{1}{(g_H+\delta)^{\frac{1}{2}}})$.
(ii) If $2(\mu+\rho) < g_H + \delta$, *then* $\hat{\sigma} \in (\frac{(\rho+\mu)^{\frac{1}{2}}}{g_H+\delta}, \frac{1}{(g_H+\delta)^{\frac{1}{2}}})$ *exist such that*
(a) $k_H^* = k_{H,L}$ *and* $c^{T*} = c_L^T$ *hold for* $\sigma = \hat{\sigma}$,
(b) $k_H^* < k_{H,L}$ *and* $c^{T*} < c_L^T$ *hold for* $\sigma \in (\frac{(\rho+\mu)^{\frac{1}{2}}}{g_H+\delta}, \hat{\sigma})$,
(c) $k_H^* > k_{H,L}$ *and* $c^{T*} > c_L^T$ *hold for* $\sigma \in (\hat{\sigma}, \frac{1}{(g_H+\delta)^{\frac{1}{2}}})$.

Proof: See Appendix O.3

Proposition A3 is consistent with Proposition 7 in the AK model. For large values of σ, the crowd-in effects of bubbles on capital accumulation dominate the crowding-out effects, and thus, bubbles promote capital accumulation, which increases output and total consumption.

O.1 Equilibrium Dynamics under Neoclassical Economy

We define the detrended output $y_{H,t} \equiv Y_t/H_t$ by $y_{H,t} = F(K_t/H_t, L) \equiv f(k_{H,t})$, where $k_{H,t} \equiv K_t/H_t$, and we use the market-clearing condition for labor $L_t = L$. Note that $q_t = \partial F/\partial K_t = f'(k_{H,t})$.

The following lemma characterizes the equilibrium dynamics in both the bubbly and bubbleless economies.

Lemma A10 *Assume that* $\sigma > 0$. *In an equilibrium in which* $I_t > 0$ *holds,* V_t, B_t, *and* $k_{H,t}$ *satisfy the following three equations:*

$$f'(k_{H,t}) = \left[\frac{\rho}{V_t} + \frac{1-V_t}{\sigma^2}\right](1+B_t), \tag{O.2}$$

$$\dot{B}_t = \left\{\mu(1+B_t) + f'(k_{H,t})V_t - \frac{1-V_t}{\sigma^2}(1+B_t)\right\}B_t, \tag{O.3}$$

$$\dot{k}_{H,t} = \left\{\frac{1-V_t}{\sigma^2}(1+B_t) - (g_H+\delta)\right\}k_{H,t}. \tag{O.4}$$

(Proof) First, we derive (O.2). The market clearing condition for general goods is given by $Y_t = C_t + C_t^w + I_t$. If $I_t > 0$, then (10c) holds. Because the general good sector is competitive, $Y_t = q_tK_t + w_tL$ holds, and thus, we obtain

$$q_tK_t = C_t + I_t,$$

where we use (N.4). As in the derivation of (15a), using (10b), (10c), (13), and $q_t = f'(k_{H,t})$, we obtain (O.2). The derivation of (O.3) is the same as that of (15b) except for $q_t = f'(k_{H,t})$.

From $k_{H,t} \equiv \frac{K_t}{H_t}$, the law of motion of $k_{H,t}$ is given by

$$\dot{k}_{H,t} = \frac{I_t}{H_t} - (g_H + \delta)k_{H,t}. \tag{O.5}$$

Substituting (10c) into the above equation and using (13) yields (O.4). Lemma A10 is proved. □

O.2 Proof of Proposition A2

First, we prove the following two lemmas, which are useful for the proof of Proposition A2.

Lemma A11 *Assume that $\sigma > 0$. Then, there exists a unique bubbleless steady-state equilibrium such that $I_t > 0$ holds and V_L and $k_{H,L}$ satisfy*

$$V_L = 1 - \sigma^2(g_H + \delta)(> 0), \tag{O.6}$$

$$q_L \equiv f'(k_{H,L}) = \frac{\rho}{1 - \sigma^2(g_H + \delta)} + g_H + \delta, \tag{O.7}$$

if and only if $1 - \sigma^2(g_H + \delta) > 0$ or, equivalently,

$$\sigma < \frac{1}{(g_H + \delta)^{\frac{1}{2}}}. \tag{O.8}$$

(Proof) With $B = 0$, solving (O.2) and (O.3) for V and $f'(k_{H,L})$ yields (O.6) and (O.7). Thus, if and only if (O.8) holds, we obtain $V_L > 0$. From (10c), $V_L \equiv 1/v_L$, and $B = 0$, we have $I_t/K_t = (1 - V_L)/\sigma^2$. This implies that $I_t > 0$ if and only if $V_L > 0$. Lemma A11 is proved. □

Lemma A12 *Assume that $\sigma > 0$. Then, there exists a unique bubbly steady-state equilibrium in which $I_t > 0$ holds and V^*, B^*, and k_H^* satisfy*

$$V^* = 1 - \sigma(\rho + \mu)^{\frac{1}{2}}(> 0), \tag{O.9}$$

$$B^* = \frac{\sigma(g_H + \delta)}{(\rho + \mu)^{\frac{1}{2}}} - 1(> 0), \tag{O.10}$$

$$f'(k_H^*) = \frac{g_H + \delta}{V^*}\left\{1 - \frac{\sigma\mu}{(\rho + \mu)^{\frac{1}{2}}}\right\}(> 0), \tag{O.11}$$

if and only if

$$\frac{(\rho + \mu)^{\frac{1}{2}}}{g_H + \delta} < \sigma < \frac{1}{(\rho + \mu)^{\frac{1}{2}}}. \tag{O.12}$$

(Proof) Suppose that $B_t > 0$. Then, (O.3), $q_t = f'(k_{H,t})$, and $\dot{B}_t = 0$ imply

$$f'(k_{H,t})V_t = \left\{ \frac{1 - V_t}{\sigma^2} - \mu \right\} (1 + B_t). \tag{O.13}$$

Multiplying both sides of (O.2) by V_t yields

$$f'(k_{H,t})V_t = \left\{ \rho + V_t \frac{1 - V_t}{\sigma^2} \right\} (1 + B_t). \tag{O.14}$$

Substituting (O.13) into (O.14) gives $V = 1 \pm \sigma(\rho + \mu)^{1/2}$. From (10c) and (13), we have $I_t = (1 - V_t)(1 + B_t)K_t/\sigma^2$. To guarantee $I_t > 0$, $V_t < 1$ must hold. Thus, we have (O.9). The dynamics of capital (O.4) imply

$$\dot{k}_{H,t} = 0 \iff \frac{1 - V_t}{\sigma^2}(1 + B_t) = g_H + \delta. \tag{O.15}$$

Substituting (O.9) into (O.15) yields (O.10). In addition, if we substitute (O.9) and (O.10) into (O.13), we obtain (O.11).

The second inequality of condition (O.12) implies $1 > \sigma(\rho + \mu)^{\frac{1}{2}}$, and thus, $V^* > 0$ holds. $0 < V^* < 1$ means that $I_t > 0$ holds. The first inequality of condition (O.12) means $\sigma(g_H + \delta) > (\rho + \mu)^{\frac{1}{2}}$, which ensures that $B^* > 0$ holds. From (F.4), $1/(\rho + \mu)^{\frac{1}{2}} < (\rho + \mu)^{\frac{1}{2}}/\mu$ holds. Then, condition (O.12) implies $(\rho + \mu)^{\frac{1}{2}} > \sigma\mu$, which ensures that $f'(k_H^*) > 0$.

Conversely, suppose that $V^* > 0$, $B^* > 0$, and $f'(k_H^*) > 0$. Note that $V^* > 0$ and $f'(k_H^*) > 0$ imply

$$\sigma < \min \left\{ \frac{1}{(\rho + \mu)^{\frac{1}{2}}}, \frac{(\rho + \mu)^{\frac{1}{2}}}{\mu} \right\}. \tag{O.16}$$

From (F.4), we have $\sigma < 1/(\rho + \mu)^{\frac{1}{2}}$. In addition, $B^* > 0$ implies $\sigma(g_H + \delta) > (\rho + \mu)^{\frac{1}{2}}$, thus, condition (O.12) holds. Lemma A12 is proved. □

We now prove Proposition A2. Note that condition (O.12) implies

$$\rho + \mu < g_H + \delta. \tag{O.17}$$

If we use (O.17), the following relationship holds:

$$\frac{(\rho + \mu)^{\frac{1}{2}}}{g_H + \delta} < \frac{1}{(g_H + \delta)^{\frac{1}{2}}} < \frac{1}{(\rho + \mu)^{\frac{1}{2}}}. \tag{O.18}$$

From (O.18), Lemma A11, and A12, we obtain Proposition A2.

O.3 Proof of Proposition A3

First, we show the following lemma.

Lemma A13 *Suppose that both bubbly and bubbleless steady states exist. We obtain*

$$f'(k_H^*) > (=)(<) f'(k_{H,L}) \iff \Omega(\sigma) < (=)(>) 0, \tag{O.19}$$

where $\Omega(\sigma) \equiv \sigma^2 + \frac{(\rho+\mu)^{\frac{1}{2}}}{g_H+\delta}\sigma - \frac{1}{g_H+\delta}$. $f'(k_{H,L})$ *and* $f'(k_H^*)$ *are given by* (O.7) *and* (O.11), *respectively.*

(Proof) Suppose that both bubbly and bubbleless steady states exist, that is, the case of (*ii*) in Proposition A2. Thus, the following condition now holds:

$$\frac{(\rho+\mu)^{\frac{1}{2}}}{g_H+\delta} < \sigma < \frac{1}{(g_H+\delta)^{\frac{1}{2}}}. \tag{O.20}$$

Recall that $f'(k_H^*)$ and $f'(k_{H,L})$ are given by

$$f'(k_H^*) = \frac{g_H+\delta}{1-\sigma(\rho+\mu)^{\frac{1}{2}}}\left(1-\frac{\sigma\mu}{(\rho+\mu)^{\frac{1}{2}}}\right),$$
$$f'(k_{H,L}) = \frac{\rho}{1-\sigma^2(g_H+\delta)} + g_H+\delta.$$

Using the above equations, we have the following relationship:

$$\begin{aligned}
&f'(k_H^*) > (=)(<) f'(k_{H,L}) \\
\iff\ & \frac{g_H+\delta}{1-\sigma(\rho+\mu)^{\frac{1}{2}}}\left(1-\frac{\sigma\mu}{(\rho+\mu)^{\frac{1}{2}}}\right) > (=)(<) \frac{\rho}{1-\sigma^2(g_H+\delta)} + g_H+\delta \\
\iff\ & \sigma^3 - \frac{(g_H+\delta+\rho+\mu)}{(g_H+\delta)^2}\sigma + \frac{(\rho+\mu)^{\frac{1}{2}}}{(g_H+\delta)^2} < (=)(>) 0 \\
\iff\ & \left(\sigma - \frac{(\rho+\mu)^{\frac{1}{2}}}{g_H+\delta}\right)\left(\sigma^2 + \frac{(\rho+\mu)^{\frac{1}{2}}}{g_H+\delta}\sigma - \frac{1}{g_H+\delta}\right) < (=)(>) 0 \\
\iff\ & \left(\sigma - \frac{(\rho+\mu)^{\frac{1}{2}}}{g_H+\delta}\right)\Omega(\sigma) < (=)(>) 0 \\
\iff\ & \Omega(\sigma) < (=)(>) 0.
\end{aligned} \tag{O.21}$$

The last line uses the first inequality of (O.20), which ensures that $\sigma - \frac{(\rho+\mu)^{\frac{1}{2}}}{g_H+\delta} > 0$. Lemma A13 is proved. □

Next, we show the following lemma.

Lemma A14 *Suppose that* (O.20) *holds.*
(i) If $2(\rho+\mu) \geq g_H+\delta$*, then* $\Omega(\sigma) > 0$ *holds for* $\sigma \in \left(\frac{(\rho+\mu)^{\frac{1}{2}}}{g_H+\delta}, \frac{1}{(g_H+\delta)^{\frac{1}{2}}}\right)$.
(ii) If $2(\rho+\mu) < g_H+\delta$*, then* $\hat{\sigma} > 0$ *exists such that*

$$\Omega(\sigma)\begin{cases} \leq 0 & \text{for } \sigma \in \left(\frac{(\rho+\mu)^{\frac{1}{2}}}{g_H+\delta}, \hat{\sigma}\right), \\ > 0 & \text{for } \sigma \in \left(\hat{\sigma}, \frac{1}{(g_H+\delta)^{\frac{1}{2}}}\right). \end{cases} \tag{O.22}$$

where $\hat{\sigma}$ *is the positive solution of* $\Omega(\sigma) = 0$.
(Proof) $\Omega(\sigma)$ has the following properties:

$$\Omega(0) = -\frac{1}{g_H+\delta} < 0, \tag{O.23}$$

$$\Omega\left(\frac{1}{(g_H+\delta)^{\frac{1}{2}}}\right) = \frac{(\rho+\mu)^{\frac{1}{2}}}{(g_H+\delta)^{\frac{3}{2}}} > 0. \tag{O.24}$$

From (O.23) and (O.24), $\Omega(\sigma) = 0$ has the following positive solution:

$$\hat{\sigma} = \frac{1}{2(g_H+\delta)}\left(\sqrt{\rho+\mu+4(g_H+\delta)} - \sqrt{\rho+\mu}\right), \quad \hat{\sigma} \in (0, \frac{1}{g_H+\delta}). \tag{O.25}$$

Thus, we obtain the following relationship:

$$\Omega(\sigma)\begin{cases} \leq 0 & \text{for } \sigma \in (0, \hat{\sigma}), \\ > 0 & \text{for } \sigma \in (\hat{\sigma}, \frac{1}{g_H+\delta}). \end{cases} \tag{O.26}$$

Recall that σ satisfies (O.20). Next, we examine whether $\hat{\sigma}$ is in $(\frac{(\rho+\mu)^{1/2}}{g_H+\delta}, \frac{1}{(g_H+\delta)^{1/2}})$. From (O.25), we have

$$\frac{(\rho+\mu)^{\frac{1}{2}}}{g_H+\delta} < (=)(>)\hat{\sigma} \iff 2(\rho+\mu) < (=)(>) g_H + \delta. \tag{O.27}$$

From, (O.20), (O.26), and (O.27), we obtain Lemma A14. □

From Lemmas A13 and A14, the results of k_H in Proposition A3 are proved.

Next, we consider the consumption effects of asset bubbles in both the bubbly and bubbleless steady states. From (O.5) and $\dot{k}_{H,t} = 0$, we obtain $I_t/H_t = (g_H+\delta)k_H$. Then, the market clearing condition for goods $Y_t = C_t + C_t^w + I_t = C_t^T + I_t$ can be rearranged as

$$\begin{aligned} c^T \equiv \frac{C_t^T}{H_t} &= \frac{Y_t}{H_t} - \frac{I_t}{H_t} \\ &= f(k_H) - (g_H+\delta)k_H. \end{aligned} \tag{O.28}$$

The second line uses $I_t/H_t = (g_H+\delta)k_H$. From (O.28), the assumptions of $f''(k_H) < 0$, and the Inada condition imply that c^T is a single-peaked function with respect to k_H. We define $\tilde{k}_H$ by $f'(\tilde{k}_H) = g_H + \delta$. Then, for $k_H < \tilde{k}_H$, c^T is an increasing function with respect to k_H. From (O.7) and (O.11), we have the following relationship:

$$f'(\tilde{k}_H) < f'(k_{H,L}),\ f'(k_H^*). \tag{O.29}$$

Because $f''(k_H) < 0$ holds, (O.29) implies

$$k_{H,L}, k_H^* < \tilde{k}_H. \tag{O.30}$$

Since c^T increases with k_H for $k_H < \tilde{k}_H$, we have

$$k_{H,L} < (=)(>) k_H^* \iff c_L^T < (=)(>) c^{T*}. \tag{O.31}$$

From Lemmas A13, A14, and (O.31), we obtain Proposition A3. □

References

[1] Barro, R. and Sala-i-Martin, X. (2004) Economic Growth. MIT Press.

[2] Stokey, N. L. (2009) The Economics of Inaction: Stochastic Control Models with Fixed Costs. Princeton University Press.

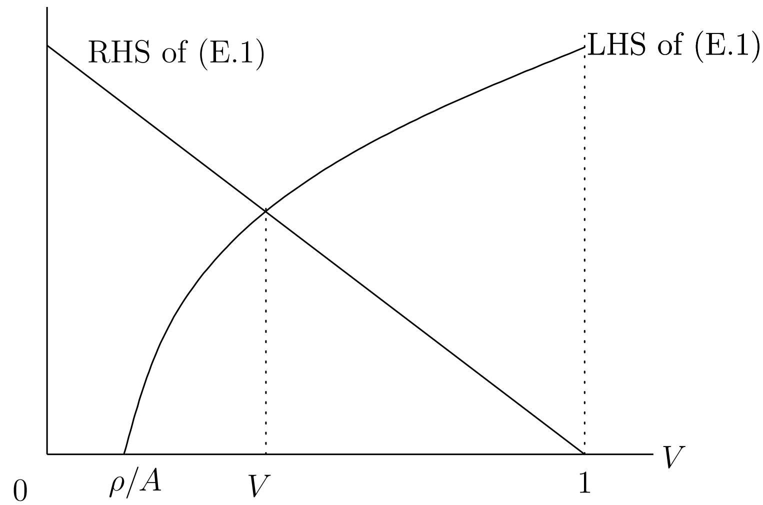

Figure A1 Bubbleless Steady State

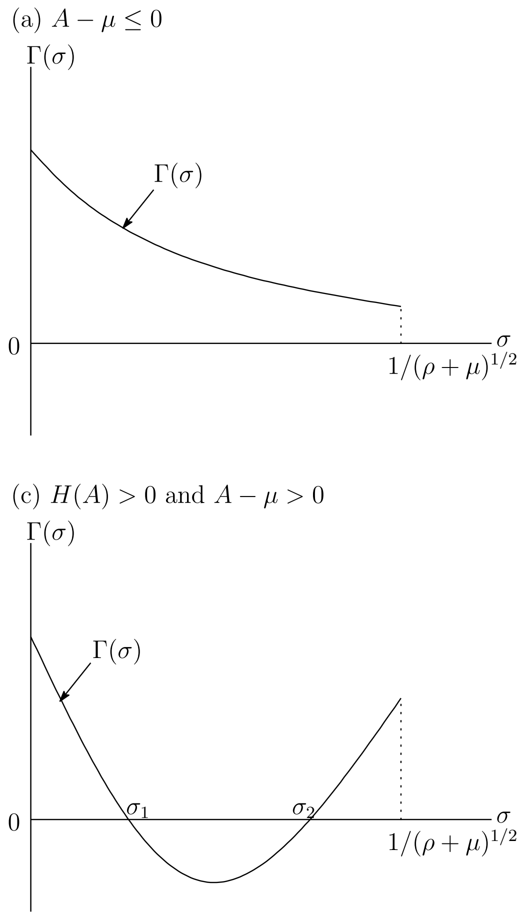

Figure A2 Function $\Gamma(\sigma)$

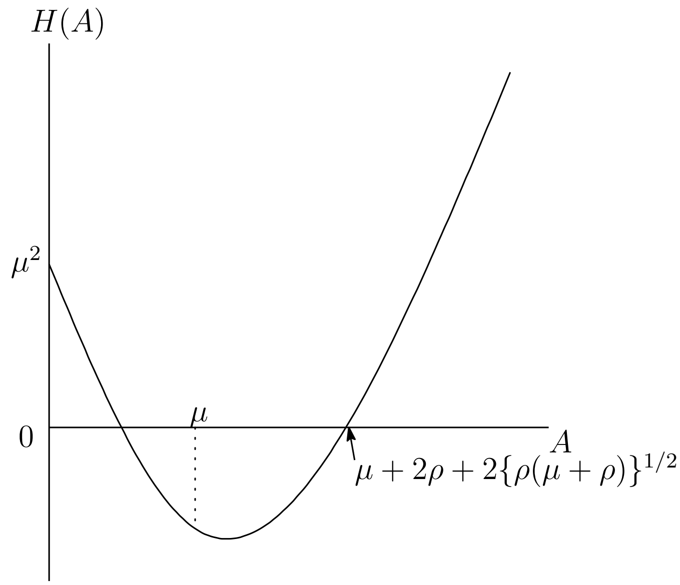

Figure A3 Function $H(A)$

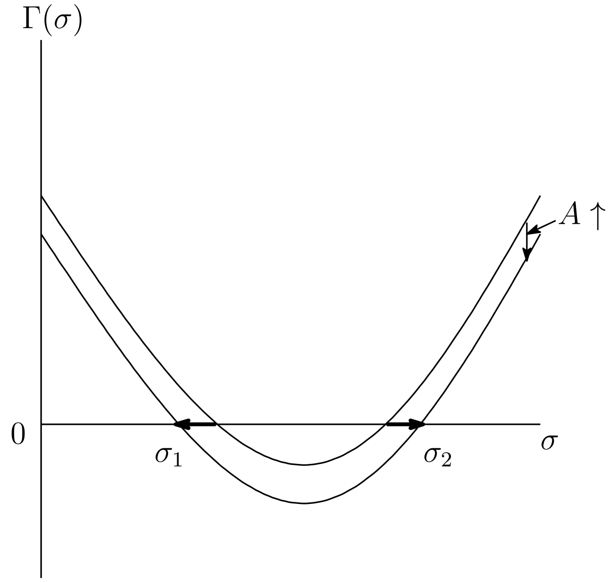

Figure A4 Proposition 6

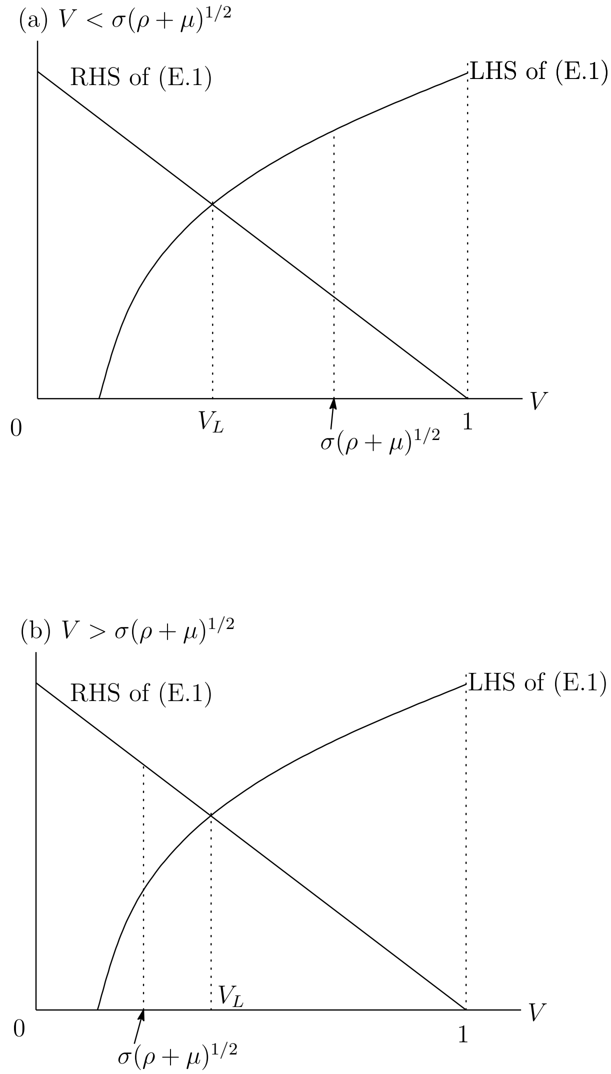

Figure A5 Relationship between V and $\sigma(\rho+\mu)^{1/2}$

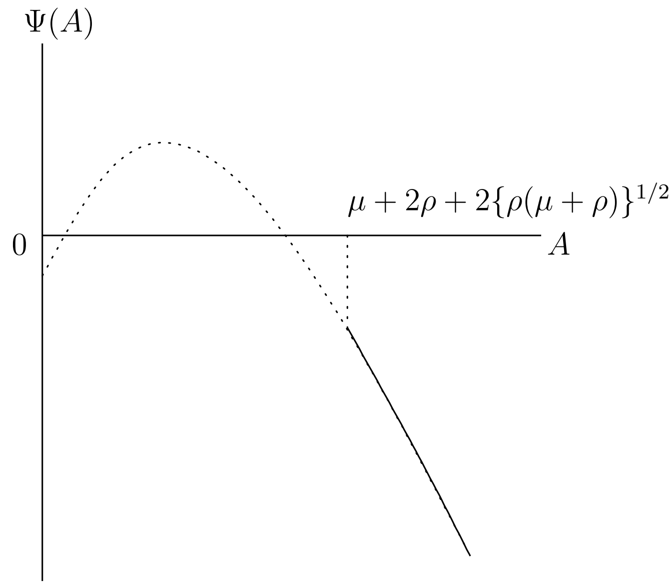

Figure A6 Function $\Psi(A)$