On Two Point Taylor Expansion

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by

Koichiro Shimada

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Preface

As is well known, if a function is analytic on an interval, then the function on a subinterval is expressed as the Taylor expansion about each point in the interval. Furthermore, possibility of Taylor expansions of functions about two or three point has been studied as useful expressions in many areas of mathematical analysis. In this thesis, for given positive integers n, m, we show possibility of two point Taylor expansions of functions about two points $-1, 1$ with multiplicity weight (n, m).

This thesis is composed of four chapters and has three main results about two point Taylor expansion.

In Chapter 1, we review important results about best approximation and interpolation by polynomials. Also, we introduce previous studies about two point Taylor expansion.

In Chapter 2, we discuss the first main theorem about two point Taylor expansion of piecewise analytic function. We show the following theorem. Let δ_1, δ_2 be real numbers with $\delta_1 > \frac{n-m}{n+m} - (-1)$ and $\delta_2 > 1 - \frac{n-m}{n+m}$, where $\frac{n-m}{n+m}$ is the point which divides the interval $[-1, 1]$ in the ratio $n : m$. Let f be a piecewise analytic function such that f is equal to an analytic function p on $[\frac{n-m}{n+m}, \infty)$ which has the Taylor expansion of p about 1 on $(1 - \delta_2, 1 + \delta_2)$, and f is equal to an analytic function q on $(-\infty, \frac{n-m}{n+m})$ which has the Taylor expansion of q about -1 on $(-1 - \delta_1, -1 + \delta_1)$. Then, it holds that f is expressed as the two point Taylor expansion about $-1, 1$ with the multiplicity weight (n, m) on the interval $[\alpha, \beta] \setminus \{\frac{n-m}{n+m}\}$, where α, β are the solutions of $|(x+1)^n (x-1)^m| = \frac{2^{n+m} \cdot n^n \cdot m^m}{(n+m)^{n+m}}$ with $\alpha < -1$ and $\beta > 1$. Also, if $p(\frac{n-m}{n+m}) = q(\frac{n-m}{n+m})$, then f is expressed as the two point Taylor expansion about $-1, 1$ with the multiplicity weight (n, m) on the interval $[\alpha, \beta]$.

In Chapter 3, we discuss the second main theorem about two point Taylor expansion of a Heaviside function. We show the following theorem. Let f be the Heaviside function such that f is equal to 1 on $[\frac{n-m}{n+m}, \infty)$, and f is equal to 0. Let $p_{f,\{-1,1\}(n\ell, m\ell)}, \ell \in \mathbf{N}$ be the Hermite interpolating polynomials to f at $-1, 1$ with multiplicities $n\ell, m\ell$. Then, there exists a positive number C such that $\left| p_{f,\{-1,1\}(n\ell,m\ell)} \left(\frac{n-m}{n+m}\right) - \frac{1}{2} \right| \leq \frac{C}{\sqrt{\ell}}$, $\ell \in \mathbf{N}$.

In Chapter 4, we discuss the third main theorem about termwise differentiation of two point Taylor expansion. We show the following theorem. Let f be a piecewise polynomial function such that f is equal to a polynomial function p of degree at most N on $[\frac{n-m}{n+m}, \infty)$, and f is equal to a polynomial function q of degree at most N on $(-\infty, \frac{n-m}{n+m})$. Then, it holds that the k-th order derivatives of f on $(\alpha, \frac{n-m}{n+m}) \cup (\frac{n-m}{n+m}, \beta)$ are expressed as the termwise k times differentiation of the two point Taylor expansion about -1, 1 with multiplicity weight (n, m).

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Contents

Chapter 1

Introduction

1.1 Polynomial approximation

As is well known, polynomial approximation has a a long history and has established the foundation of approximation theory. Specially, best approximation and interpolation by polunomials play important roles of polynomial approximation and have been furnishing us with challenging topics and problems. Before making a brief review of best approximation and interpolation by polynomials, we give some notations and definitions.

Notation 1.1.1. (1) Let $[a, b]$ $(-\infty < a < b < \infty)$ be a real compact interval and $C[a, b]$ the space of all real-valued continuous functions on $[a, b]$.
(2) $\|\cdot\|_\infty$ denotes the supremum norm on $C[a, b]$, i.e.,

$$\|f\|_\infty = \sup_{x\in[a,b]} |f(x)|, \qquad f \in C[a, b].$$

(3) For each nonnegative integer n, $\mathcal{P}_n$ express the space of polynomials of degree at most n.

Definition 1.1.2. For any $f \in C[a, b]$, there exists a unique polynomial $p^* \in \mathcal{P}_n$ such that

$$\|f - p^*\|_\infty \le \|f - p\|_\infty \quad \text{for all } p \in \mathcal{P}_n.$$

The polynomial p^* is called the *best uniform approximation to f from $\mathcal{P}_n$* (or simply the *best uniform approximation to f*).

It is well known that any continuous functions can be approximated by polynomial functions (Weierstrass(1885)).

Theorem 1.1.3. *For any given $f \in C[a, b]$ and any $\varepsilon > 0$, there exists a polynomial p such that*

$$\|f - p\|_\infty < \varepsilon.$$

The Russian mathematician P. L. Chebyshev studied best uniform approximation from $\mathcal{P}_n$ to a function in $C[a, b]$.

Theorem 1.1.4 (Kincaid and Cheney [9, Corollary 6 in p. 416])**.** *Let $f \in C[a,b]$. In order that $p_n \in \mathcal{P}_n$ is the best uniform approximation to f, it is necessary and sufficient that there exist $(n+2)$ points $x_0, \ldots, x_{n+1}(x_0 < \cdots < x_{n+1})$ in $[a,b]$ and $\sigma = 1$ or -1 such that*

$$f(x_i) - p(x_i) = \sigma(-1)^i \|f - p\|_\infty, \qquad 0 \le i \le n+1.$$

From Theorem 1.1.3 and Theorem 1.1.4, we easily have the following.

Theorem 1.1.5. *For any given $f \in C[a,b]$, let $p_n, n \in \mathbf{N}$ be the best uniform approximation to f from $\mathcal{P}_n$. Then, it holds that $\|f - p_n\|_\infty \to 0$ $(n \to \infty)$.*

1.2 Lagrange interpolating polynomials

In the rest of this chapter, we review important results about interpolation by polynomials. In 1.2, some results about Lagrange interpolating polynomials are stated and we show several results about Hermite interpolating polynomials, in particular, results about two point Taylor expansions.

First, we begin with the definition of interpolation by polynomials.

Definition 1.2.1. Let I be an infinite subset of $\mathbf{R}$ and f a real-valued function on I. For any given finite subset $X = \{x_0, \ldots, x_n\}$ of I and for any given positive integers $k_0, \ldots, k_n$, if the values of the derivatives $f^{(j)}(x_i), 0 \le i \le n, 0 \le j \le k_i - 1$ exist, then there exists a unique approximating polynomial $p_{f,X(k_0,\ldots,k_n)}(x)$ to f which is of degree at most $m(= k_0 + \cdots + k_n - 1)$ and satisfies that

$$p^{(j)}_{f,X(k_0,\ldots,k_n)}(x_i) = f^{(j)}(x_i), \qquad 0 \le i \le n,\ 0 \le j \le k_i - 1.$$

The points $x_0, \ldots, x_n$ and the polynomial $p^{(j)}_{f,X(k_0,\ldots,k_n)}(x)$ are called *nodes* and the *Hermite interpolating polynomial to f at $x_0, \ldots, x_n$ with multiplicities* $k_0, \ldots, k_n$, respectively. In particular, if $k_0 = \cdots = k_n = 1$, we simply write $p_{f,X}(x)$ for $p_{f,X(1,\ldots,1)}(x)$ and call it the *Lagrange interpolating polynomial to f at $x_0, \ldots, x_n$*.

For any $f \in C[a,b]$, let $p_n \in \mathcal{P}_n, n \in \mathbf{N}$ be the best uniform approximation to f. From Theorem 1.1.4, since $f - p_n$ has at least $(n+1)$ zeros in $[a,b]$, we put a set $X_n = \{x^{(n)}_0, \ldots, x^{(n)}_n\}, n \in \mathbf{N}$ consisting $(n+1)$ points of $\{x \mid f(x) - p_n(x) = 0, x \in [a,b]\}$. Then we immediately have the following.

Theorem 1.2.2. *For any $f \in C[a,b]$, let $X_n, n \in \mathbf{N}$ be the finite subsets of $[a,b]$ stated above. Then, it holds that $\|f - p_{f,X_n}\|_\infty \to 0$ $(n \to \infty)$.*

On the other hand, Runge[18] and Bernstein[1] showed the results which tell us the importance of selecting appropriate nodes.

Theorem 1.2.3. *Let $f(x) = \dfrac{1}{1+25x^2}$ and $g(x) = |x|, x \in [-1,1]$ and let*

$$X_n = \left\{ x^{(n)}_i = -1 + \frac{2i}{n} \;\middle|\; 0 \le i \le n \right\}, \quad n \ge 1$$

the sequence of systems of equidistant nodes in $[-1,1]$*. Then, it holds that*

$$\lim_{n\to\infty} \|f - p_{f,X_n}\|_\infty = +\infty$$

and

$$\limsup_{n\to\infty} |g(x) - p_{f,X_n}(x)| = +\infty \quad \textit{for every } x \in (-1,1)\setminus\{0\}.$$

To explain possibility of approximation by Lagrange interpolating polynomials, we make a definition of Lagrange interpolation operator from $C[-1,1]$ to $C[-1,1]$.

Definition 1.2.4. Let X be a subset of $[-1,1]$ consisting of $(n+1)$ nodes $x_0,\dots,x_n$ ($x_0 < \cdots < x_n$). We put

$$\ell_i(x) = \frac{(x-x_0)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}, \quad i = 0,\dots,n.$$

For any given $f \in C[-1,1]$, the Lagrange interpolating polynomial $p_{f,X}(x)$ is expressed as $\sum_{i=0}^n f(x_i)\ell_i(x)$. Then, we set a linear operator L from $C[-1,1]$ to $C[-1,1]$ such that

$$L(f) = \sum_{i=0}^{n} f(x_i)\ell_i(x), \qquad f \in C[-1,1]$$

and the linear operator L is called the *Lagrange interpolation operator at* $x_0,\dots,x_n$.

When we consider a bounded linear operator L from $(C[-1,1], \|\cdot\|_\infty)$ to $(C[-1,1], \|\cdot\|_\infty)$, the norm of L is denoted by $\|L\|_\infty$. Lagrange interpolation operators from $(C[-1,1], \|\cdot\|_\infty)$ to $(C[-1,1], \|\cdot\|_\infty)$ are bounded and the following results about norms of Lagrange interpolation operators are well known.

Theorem 1.2.5 (Nürnberger [16, p. 27])**.** *For a Lagrange interpolation operator* L *at nodes* $x_0,\dots,x_n$ *in* $[-1,1]$*, it holds that*

$$\|L\|_\infty = \left\| \sum_{i=0}^{n} |\ell_i(x)| \right\|_\infty .$$

Theorem 1.2.6 (Rivlin [17, p. 23])**.** *For a Lagrange interpolation operator* L *at nodes* $x_0,\dots,x_n$ $(n \geq 2)$ *in* $[-1,1]$*, it holds that*

$$\|L\|_\infty > \frac{2}{\pi}\log(n+1) + \frac{1}{2}.$$

Let us consider any sequence of system $\{x_0^{(n)},\dots,x_n^{(n)}\}, n \geq 1$ of nodes in $[-1,1]$ and $L_n, n \geq 1$ the Lagrange interpolation operators at nodes $x_0^{(n)},\dots,x_n^{(n)}$. By Theorem 1.2.6, there exist an $f \in C[-1,1]$ such that

$$\limsup_{n\to\infty} \|f - L_n f\|_\infty = +\infty.$$

Hence, there exists no good sequnece of system $\{x_0^{(n)},\dots,x_n^{(n)}\}, n \geq 1$ of nodes in $[-1,1]$ satisfying that

$$\lim_{n\to\infty} \|f - L_n f\|_\infty = 0 \text{ for all } f \in C[-1,1].$$

But the minimum of norms of Lagrange interpolation operators has been profoundly studied. For given $(n+1)$ nodes $x_0, \dots, x_n$ $(x_0 < \cdots < x_n)$ in $[-1,1]$, we call the function $\lambda(x; x_0, \dots, x_n) := \sum_{i=0}^{n} |\ell_i(x)|$ in Theorem 1.2.5 the *Lebesgue function* and write $M_i(x_0, \dots, x_n)$ for the maximum of $\lambda(x; x_0, \dots, x_n)$ on $[x_{i-1}, x_i], i = 1, \dots, n$. Bernstein [2] and Erdös [6] conjectured the following neccesary and sufficient condition under which norms of Lagrange interpolation operator is minimized.

Conjectures by Bernstein and Erdös. Let $x_0, \dots, x_n$ $(-1 = x_0 < \cdots < x_n = 1)$ be nodes in $[-1,1]$. The norm of the Langrange interpolation operator is minimum at $x_0, \dots, x_n$ if and only if

$$M := M_1(x_0, \dots, x_n) = \cdots = M_n(x_0, \dots, x_n). \qquad (*)$$

Nodes which satisfy $(*)$ are uniquely detemined and for any nodes $z_0, \dots, z_n$ $(-1 = z_0 < \cdots < z_n = 1)$, it holds that

$$\min_{i=1,\dots,n} M_i(z_0, \dots, z_n) \le M.$$

The conjectures stated above had not been proven for nearly 50 years, but Kilgore [8] and de Boor and Pinkus [3] independently obtained proofs of the conjectures.

Let $\|\cdot\|_I$ be the norm on $C[a,b]$ such that

$$\|f\|_I := \sup_{[\alpha,\beta]\subset[a,b]} \left| \int_\alpha^\beta f(x)\,dx \right|, \qquad f \in C[a,b]$$

and $\|L\|_I$ the norm of a Lagrange interpolation operator from $(C[-1,1], \|\cdot\|_\infty)$ to $(C[-1,1], \|\cdot\|_I)$. Then, a conjecture of the minimum of norms $\|L\|_I$ of Lagrange interpolation operators is stated in Kitahara[11].

Conjecture about $\|L\|_I$. For a given Lagrange interpolation operator L at $x_0, \dots, x_n$ $(-1 \le x_0 < \cdots < x_n \le 1)$, $\|L\|_I$ is minimum if and only if

$$\|L\|_I = \sum_{i=0}^{n} \left| \int_{-1}^{1} \ell_i(x)\,dx \right| = 2.$$

1.3 Hermite interpolating polynomials

Hermite interpolating polynomials much concern expansions of functions. Let f be a sufficiently differentiable function and consider a one point x_0 as one node and set $X = \{x_0\}$. Then the Hermite interpolating polynomial $p_{f,X(n)}$ to f at x_0 with multiplicity n is the Taylor polynomial of f about x_0, that is

$$p_{f,X(n)} = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \cdots + \frac{f^{(n-1)}(x_0)}{(n-1)!}(x - x_0)^{n-1}.$$

Furthermore, if f is infinitely differentiable at x_0 and if

$$f(x) = \lim_{n\to\infty} p_{f,X(n)}(x) \quad \text{for all } x \in (x_0 - \rho, x_0 + \rho)\ (\rho > 0),$$

then f has the Taylor expansion of f at x_0 on $(x_0 - \rho, x_0 + \rho)$. From this, if X is a finite set, then we can make the following definition.

Definition 1.3.1. Let f be a real-valued function on a subset A of the real line $\mathbf{R}$ whose interior is not empty. Let $X = \{x_0, \ldots, x_{m-1}\}$ be a set of m distinct nodes in the interior of A such that f is infinitely differentiable at $x_0, \ldots, x_{m-1}$. For given positive integers $w_0, \ldots, w_{m-1}$, if

$$\lim_{n\to\infty} p_{f,X(w_0 n,\ldots,w_{m-1}n)}(x) = f(x) \quad \text{for all} \quad x \in A,$$

then we say that f has *the m point Taylor expansion about $x_0, \ldots, x_{m-1}$ with multiplicity weight $(w_0, \ldots, w_{m-1})$ on A.*

The notion of two point or m point Taylor expanson is not new and Taylor expansions of functions about two or three point has been studied as much useful expression in mathematical analysis.

Representations of $p_{f,X(n,\ldots,n)}(x)$ are seen in Davis [4, p. 37].

Theorem 1.3.2. *Let f be a sufficiently differentiable at two points a and b and let $X = \{a, b\}$. For a given positive integer n,*

$$p_{f,X(n,n)}(x) = (x-a)^n \sum_{k=0}^{n-1} \frac{B_k(x-b)^k}{k!} + (x-b)^n \sum_{k=0}^{n-1} \frac{A_k(x-a)^k}{k!},$$

where $A_k = \dfrac{d^k}{dx^k}\left[\dfrac{f(x)}{(z-b)^n}\right]_{x=a}$ *and* $B_k = \dfrac{d^k}{dx^k}\left[\dfrac{f(x)}{(z-a)^n}\right]_{x=b}$, $k = 0, \ldots, n-1$.

In the report of Estes and Lancaster [7], a comparison of the resulting solutions for the two-body problem from the two point Taylor expansions and one point Taylor expansions is shown. In the book by Walsh [21, Chap. 3], we can see several results on m point Taylor expansion of analytic functions on and within lemniscates of the complex plane. By Theorem 1 in López and Temme [15], we can give the following result of two point Taylor expansions of analytic functions on a simply connected domain of the complex plane $\mathbf{C}$.

Theorem 1.3.3. *Let $f(z)$ be an anlytic function on a simply connected domain $\Omega \subset \mathbf{C}$ and $z_1, z_2 \in \Omega$ with $z_1 \neq z_2$. Let $O_{z_1,z_2} = \{z \in \Omega \mid |(z-z_1)(z-z_2)| < r\}$, where $r = \inf_{w\in\mathbf{C}-\Omega}\{|(w-z_1)(w-z_2)|\}$. Then, $f(z)$ admits the two point Taylor expansion*

$$f(z) = \sum_{n=0}^{\infty} [a_n(z_1,z_2)(z-z_1) + a_n(z_2,z_1)(z-z_2)](z-z_1)^n(z-z_2)^n, \; z \in O_{z_1,z_2},$$

where

$$a_n(z_1,z_2) = \frac{1}{2\pi i(z_2-z_1)} \int_C \frac{f(w)\,dw}{(w-z_1)^n(w-z_2)^{n+1}}, \qquad n = 0, 1, 2, \ldots$$

and C is a simple closed loop which encircles the points z_1 and z_2 in the counterclockwise direction and is contained in Ω.

Furthermore, López and Sinusía [14] considered the boundary value problem

$$\begin{cases} \varphi(x)y'' + f(x)y' + g(x)y = h(x) \text{ in } (-1,1) \\ B\begin{pmatrix} y(-1) \\ y(1) \\ y'(-1) \\ y'(1) \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \end{cases}$$

where $\varphi(x), f(x), g(x)$ and $h(x)$ are analyitc in a Cassini disk with foci at $x = \pm 1$ containing the interval $(-1, 1)$ and $\alpha, \beta \in \mathbf{R}$ and B is a 2×4 matrix of rank 2 which defines the Dirichlet, Neumann or mixed Dirichlet-Neumann boundary conditions. In order to give a criterion for the existence and uniqueness of solution of this boundary value problem, the two point Taylor expansion of the solution $y(x)$ about the extreme points ± 1 is used.

As another point of view of two point Taylor expansion, Kitahara et al [10, 13, 12], Shimada [19] and Taguchi [20] have interesting discussions on possibility of two point Taylor expansions of functions on a real interval which are not always analytic.

Theorem 1.3.4 (Kitahara, Chiyonobu and Tsukamoto [10, Theorem]). *Let f be a function on $\mathbf{R}$, which is expressed as*

$$f(x) = \begin{cases} p(x) & x \in [0, \infty) \\ q(x) & x \in (-\infty, 0) \end{cases},$$

where p and q are polynomials of degree at most n. Let $p_{f,\{-1,1\}(\ell,\ell)}, \ell \in \mathbf{N}$ be the Hermite interpolating polynomials to f at $-1, 1$ with multiplicities ℓ, ℓ. Then, f has the two point Taylor expansion about $-1, 1$ with multiplicity weight $(1, 1)$ on $\left(-\sqrt{2}, 0\right) \cup \left(0, \sqrt{2}\right)$, that is,

$$\lim_{\ell\to\infty} p_{f,\{-1,1\}(\ell,\ell)}(x) = f(x) \text{ for all } x \in \left(-\sqrt{2}, 0\right) \cup \left(0, \sqrt{2}\right).$$

Moreover, if $p(0) = q(0)$, then f has the two point Taylor expansion about $-1, 1$ with multiplicity weight $(1, 1)$ on $\left(-\sqrt{2}, \sqrt{2}\right)$, that is,

$$\lim_{\ell\to\infty} p_{f,\{-1,1\}(\ell,\ell)}(x) = f(x) \text{ for all } x \in \left(-\sqrt{2}, \sqrt{2}\right).$$

Theorem 1.3.5 (Kitahara, Yamada and Fujiwara [13, Theorem 3]). *Let f be a real-valued function on $\mathbf{R}$ which is expressed as*

$$f(x) = \begin{cases} C_1 & x \in [0, \infty) \\ C_2 & x \in (-\infty, 0) \end{cases},$$

where C_1 and C_2 are real numbers. Let $p_{f,\{-1,1\}(\ell,\ell)}, \ell \in \mathbf{N}$ be the Hermite interpolating polynomials to f at $-1, 1$ with multiplicities ℓ, ℓ. Then, it holds that

$$p_{f,\{-1,1\}(\ell,\ell)}(0) = \frac{C_1 + C_2}{2}, \ell \in \mathbf{N}.$$

Theorem 1.3.6 (Kitahara, Yamada and Fujiwara [13, Theorem 4]). *Let f be a real-valued function on $[-r, r]$ $(r > 1 + \sqrt{2})$ which is expressed as*

$$f(x) = \begin{cases} \alpha(x) & x \in [0, r] \\ \beta(x) & x \in [-r, 0) \end{cases},$$

where α (resp. β) is expressed as the Taylor expansion of α (resp. β) about 1 (resp. -1). Let P_ℓ, $\ell \in \mathbf{N}$ be the Hermite interpolating polynomials to f at $-1, 1$ with multiplicities ℓ, ℓ. Then, it holds that, for any given positive integer k

$$\lim_{\ell\to\infty} P_\ell^{(k)}(x) = f^{(k)}(x) \text{ for all } x \in \left(-\sqrt{2}, 0\right) \cup \left(0, \sqrt{2}\right).$$

Theorem 1.3.7 (Kitahara and Okuno [12, Theorem 2]). *Let f be a function on $\mathbf{R}$, which is expressed as*

$$f(x) = \begin{cases} p(x) & x \in \left[\frac{1}{3}, \infty\right) \\ q(x) & x \in \left(-\infty, \frac{1}{3}\right) \end{cases},$$

where p and q are polynomials of degree at most n. Let $p_{f,\{-1,1\}(2\ell,\ell)}, \ell \in \mathbf{N}$ be the Hermite interpolating polynomials to f at $-1, 1$ with multiplicities $2\ell, \ell$. Let α be the real number with $\alpha < -1$ and $(\alpha+1)^2(\alpha-1) = -\frac{32}{27}$ and β the real number with $\beta > 1$ and $(\beta+1)^2(\beta-1) = \frac{32}{27}$. Then, for each $x \in \left(\alpha, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \beta\right)$, there exists a positive number C

$$|p_{f,\{-1,1\}(2\ell,\ell)}(x) - f(x)| \le \frac{C}{\sqrt{\ell}} \text{ for all } \ell \in \mathbf{N},$$

that is, f has the two point Taylor expansion about $-1, 1$ with multiplicity weight $(2,1)$ on $\left(\alpha, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \beta\right)$. Moreover, if $p\left(\frac{1}{3}\right) = q\left(\frac{1}{3}\right)$, then there exists a positive number C such that

$$\left|p_{f,\{-1,1\}(2\ell,\ell)}\left(\frac{1}{3}\right) - f\left(\frac{1}{3}\right)\right| \le \frac{C}{\sqrt{\ell}}, \ell \in \mathbf{N},$$

that is, f has the two point Taylor expansion about $-1, 1$ with multiplicity weight $(2,1)$ on (α, β).

Theorem 1.3.8 (Shimada [19]). *Let m, n be positive integers. Let f be a piecewise polynomial function*

$$f(x) = \begin{cases} p(x) & x \in \left[\frac{n-m}{n+m}, \infty\right) \\ q(x) & x \in \left(-\infty, \frac{n-m}{n+m}\right) \end{cases}$$

such that p and q are polynomials of degree at most k. Let $p_{f,\{-1,1\}(n\ell,m\ell)}, \ell \in \mathbf{N}$ be the Hermite interpolating polynomials to f at $-1, 1$ with multiplicities $n\ell, m\ell$. Let α be the real number with $\alpha < -1$ and $|(\alpha+1)^n(\alpha-1)^m| = \frac{2^{n+m} \cdot n^n \cdot m^m}{(n+m)^{n+m}}$ and β the real number with $\beta > 1$ and $|(\beta+1)^n(\beta-1)^m| = \frac{2^{n+m} \cdot n^n \cdot m^m}{(n+m)^{n+m}}$. Then, for each $x \in \left[\alpha, \frac{n-m}{n+m}\right) \cup \left(\frac{n-m}{n+m}, \beta\right]$, there exists a positive number C such that

$$\left|p_{f,\{-1,1\}(n\ell,m\ell)}(x) - f(x)\right| \le \frac{C}{\sqrt{\ell}} \text{ for all } \ell \in \mathbf{N},$$

that is, f has the two point Taylor expansion about $-1, 1$ with multiplicity weight (n,m) on $\left[\alpha, \frac{n-m}{n+m}\right) \cup \left(\frac{n-m}{n+m}, \beta\right]$. In addition, for all real numbers a, b with $\alpha < a < \frac{n-m}{n+m} < b < \beta$, the sequence of functions $\{p_{f,\{-1,1\}(n\ell,m\ell)}\}_{\ell\in\mathbf{N}}$ converges to f uniformly on $[\alpha, a] \cup [b, \beta]$. Moreover, if $p\left(\frac{n-m}{n+m}\right) = q\left(\frac{n-m}{n+m}\right)$, then there exists a positive number C such that

$$\left|p_{f,\{-1,1\}(n\ell,m\ell)}\left(\frac{n-m}{n+m}\right) - f\left(\frac{n-m}{n+m}\right)\right| \le \frac{C}{\sqrt{\ell}}, \ell \in \mathbf{N},$$

that is, f has the two point Taylor expansion about $-1, 1$ with multiplicity weight (n,m) on $[\alpha, \beta]$.

Theorem 1.3.9 (Taguchi [20]). *Let m, n be positive integers. Let f be a real-valued function on $\mathbf{R}$ which is expressed as*

$$f(x) = \begin{cases} C_1 & x \in \left[\frac{n-m}{n+m}, \infty\right) \\ C_2 & x \in \left(-\infty, \frac{n-m}{n+m}\right) \end{cases},$$

where C_1 and C_2 are real numbers. Let $p_{f,\{-1,1\}(\ell,\ell)}, \ell \in \mathbf{N}$ be the Hermite interpolating polynomials to f at $-1, 1$ with multiplicities $n\ell, m\ell$. Then, it holds that

$$\lim_{\ell\to\infty} p_{f,\{-1,1\}(n\ell,m\ell)}\left(\frac{n-m}{n+m}\right) = \frac{C_1+C_2}{2}.$$

There are three purposes of this thesis. The first purpose is to show a generalization of Theorem 1.3.8 (see Chapter 2). The second purpose is to give another proof of Theorem 1.3.9 (see Chapter 3). The third purpose is to show a generalization of Theorem 1.3.6 (see Chapter 4).

Chapter 2

Two point Taylor expansion of piecewise analytic function

2.1 Main Results

The purpose of this chapter is to prove the following theorem.

Theorem 2.1.1. *Let m, n be positive integers. Let δ_1 be a real number with $\delta_1 > \frac{n-m}{n+m} - (-1)$ and δ_2 a real number with $\delta_2 > 1 - \frac{n-m}{n+m}$, where $\frac{n-m}{n+m}$ is the point which divides the interval $[-1, 1]$ in the ratio $n : m$. Let f be a piecewise analytic function*

$$f(x) = \begin{cases} p(x) & x \in \left[\frac{n-m}{n+m}, \infty\right) \\ q(x) & x \in \left(-\infty, \frac{n-m}{n+m}\right) \end{cases}$$

such that f is equal to an analytic function p on $\left[\frac{n-m}{n+m}, \infty\right)$ which has the Taylor expansion of p about 1 on $(1-\delta_2, 1+\delta_2)$, and f is equal to an analytic function q on $\left(-\infty, \frac{n-m}{n+m}\right)$ which has the Taylor expansion of q about -1 on $(-1-\delta_1, -1+\delta_1)$. Let $p_{f,\{-1,1\}(n\ell,m\ell)}$, $\ell \in \mathbf{N}$ be the Hermite interpolating polynomials to f at $-1, 1$ with multiplicities $n\ell, m\ell$. Let α be the real number with $\alpha < -1$ and $|(\alpha+1)^n(\alpha-1)^m| = \frac{2^{n+m} \cdot n^n \cdot m^m}{(n+m)^{n+m}}$ and β the real number with $\beta > 1$ and $|(\beta+1)^n(\beta-1)^m| = \frac{2^{n+m} \cdot n^n \cdot m^m}{(n+m)^{n+m}}$. Then, the following propositions hold:
(1) For each $x \in \left[\alpha, \frac{n-m}{n+m}\right) \cup \left(\frac{n-m}{n+m}, \beta\right]$, there exists a positive number C such that

$$\left|p_{f,\{-1,1\}(n\ell,m\ell)}(x) - f(x)\right| \leq \frac{C}{\sqrt{\ell}} \text{ for all } \ell \in \mathbf{N},$$

that is, f has the two point Taylor expansion about $-1, 1$ with multiplicity weight (n, m) on $\left[\alpha, \frac{n-m}{n+m}\right) \cup \left(\frac{n-m}{n+m}, \beta\right]$.
(2) For any real numbers a, b with $\alpha < a < \frac{n-m}{n+m} < b < \beta$, the sequence of functions $\{p_{f,\{-1,1\}(n\ell,m\ell)}\}_{\ell\in\mathbf{N}}$ uniformly converges to f on $[\alpha, a] \cup [b, \beta]$.
(3) If $p\left(\frac{n-m}{n+m}\right) = q\left(\frac{n-m}{n+m}\right)$, then there exists a positive number C such that

$$\left|p_{f,\{-1,1\}(n\ell,m\ell)}\left(\frac{n-m}{n+m}\right) - f\left(\frac{n-m}{n+m}\right)\right| \leq \frac{C}{\sqrt{\ell}}\ ,\ \ell \in \mathbf{N},$$

that is, f has the two point Taylor expansion about $-1, 1$ with multiplicity weight (n, m) on $[\alpha, \beta]$.

2.2 Estimation of the absolute values of divided differences

First, we review the definition of divided differences and give three necessary propositions.

Definition 2.2.1. Let $x_0, \ldots, x_n$ be a list of nodes. In the list of nodes, only distinct nodes $z_0, \ldots, z_j$ appear and each node $z_i, i = 0, \ldots, j$ is just appeared k_i times. Let f be sufficiently differentiable at $z_0, \ldots, z_j$. Let p be the Hermite interpolating polynomials to f at $z_0, \ldots, z_j$ with multiplicities $k_0, \ldots, k_j$. Then, we call the coefficient of x^n of the polynomial p is called the *n-th order divided difference of* f *at* $x_0, \ldots, x_n$ and it is denoted by $f[x_0, \ldots, x_n]$. To make sure of multiplicities, we express

$$f[z_0, \ldots, z_j; k_0, \ldots, k_j]$$

for the divided difference $f[x_0, \ldots, x_n]$.

Proposition 2.2.2 (Kincaid and Cheney [9, p. 346])**.** *Let $x_0, \ldots, x_n$ be a list of nodes and let f be a sufficiently differentiable function at $x_0, \ldots, x_n$. If p is the Hermite interpolating polynomial of f at $x_0, \ldots, x_n$, then p is expressed as*

$$p(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, \ldots, x_k](x - x_0) \cdots (x - x_{k-1}).$$

From Theorem 3 in Kincaid and Cheney[9, p. 333], we easily have the following.

Proposition 2.2.3. *Let $x_0, \ldots, x_n$ be a list of nodes and let f be a real-valued function on an interval $[a, b]$ which is sufficiently differentiable at $x_0, \ldots, x_n$. If p is the Hermite interpolating polynomial of f at $x_0, \ldots, x_n$, then*

$$f(x) - p(x) = f[x_0, \ldots, x_n, x](x - x_0)(x - x_1) \cdots (x - x_n) \ , \ x \in [a, b].$$

Proposition 2.2.4 (Kincaid and Cheney [9, p. 347])**.** *Let $z_0, \ldots, z_j$ be a list of distinct nodes and $k_0, \ldots, k_j$ positive integers. Let $x_0, \ldots, x_n$ be a list of nodes which satisfy that each node $z_i, i = 0, \ldots, j$ is just appeared k_i times like this:*

$$(x_0, \ldots, x_n) = (\underbrace{z_0, \ldots, z_0}_{k_0}, \ldots, \underbrace{z_j, \ldots, z_j}_{k_j}).$$

If a function f is sufficiently differentiable at $z_0, \ldots, z_j$, then the divided differences of f obey the following recursive formula:

$$f[x_0, \ldots, x_k] = \begin{cases} \dfrac{f[x_1, \ldots, x_k] - f[x_0, \ldots, x_{k-1}]}{x_k - x_0} & (x_k \neq x_0) \\[2ex] \dfrac{f^{(k)}(x_0)}{k!} & (x_k = x_0) \end{cases} , \ k = 0, \ldots, n.$$

Next, we need to prepare propositions to show Theorem 2.1.1.

Proposition 2.2.5. *Let M, N be positive integers. Let f be a real-valued function on* $\mathbf{R}$ *which is sufficiently differentiable at $-1, 1$. Then, the following inequality holds:*

$$\begin{aligned}&|f[-1,t,1;N,1,M]| \\ &\le \frac{1}{2^{N+M}}\binom{N+M}{M}\left(\sum_{k=1}^{N}\left(\frac{2N}{N+M}\right)^k |f[-1,t;k,1]| + \sum_{k=1}^{M}\left(\frac{2M}{N+M}\right)^k |f[t,1;1,k]|\right).\end{aligned}$$

Proof. First, we show that for any positive integers M, N,

$$\begin{aligned} f[-1,t,1;N,1,M] &= \sum_{k=1}^{N}\frac{(-1)^M}{2^{N+M-k}}\binom{N+M-(k+1)}{M-1} f[-1,t;k,1] \\ &\quad + \sum_{k=1}^{M}\frac{(-1)^{M-k}}{2^{N+M-k}}\binom{N+M-(k+1)}{N-1} f[t,1;1,k]. \end{aligned} \tag{$*$}$$

We prove this by induction. Suppose that $N = M = 1$. Then we have

$$f[-1,t,1;1,1,1] = \frac{f[t,1;1,1] - f[-1,t;1,1]}{2},$$

which is equal to the right hand formula of $(*)$.

Next, under the condition that $(*)$ hold for $N = 1$ and $M = m$, we consider the case $N = 1, M = m+1$. We obtain

$$\begin{aligned} f[-1,t,1;1,1,m+1] &= \frac{f[t,1;1,m+1] - f[-1,t,1;1,1,m]}{2} \\ &= \frac{1}{2}f[t,1;1,m+1] - \frac{1}{2}\frac{(-1)^m}{2^{1+m-1}}\binom{1+m-2}{m-1} f[-1,t;1,1] \\ &\quad -\frac{1}{2}\sum_{k=1}^{m}\frac{(-1)^{m-k}}{2^{1+m-k}}\binom{1+m-(k+1)}{0} f[t,1;1,k] \\ &= \frac{(-1)^{m+1}}{2^{1+(m+1)-1}}\binom{1+(m+1)-2}{m} f[-1,t;1,1] \\ &\quad +\sum_{k=1}^{m+1}\frac{(-1)^{(m+1)-k}}{2^{1+(m+1)-k}}\binom{1+(m+1)-(k+1)}{0} f[t,1;1,k], \end{aligned}$$

which is equal to the right hand formula of $(*)$. Hence, in an analogous way to the above, we show that $(*)$ hold for the cases that $N = 1$, M is any positive integer or the cases that N is any positive integer, $M = 1$.

Finally, under the condition that $(*)$ hold for the cases $N + M \le m + n$, we consider the case $N = n+1, M = m$. From this assumption, we get

$$\begin{aligned}
&f[-1,t,1;n+1,1,m]\\
&= \frac{f[-1,t,1;n,1,m] - f[-1,t,1;n+1,1,m-1]}{2}\\
&= \frac{1}{2}\left(\sum_{k=1}^{n} \frac{(-1)^m}{2^{n+m-k}}\tbinom{n+m-(k+1)}{m-1} f[-1,t;k,1] + \sum_{k=1}^{m} \frac{(-1)^{m-k}}{2^{n+m-k}}\tbinom{n+m-(k+1)}{n-1} f[t,1;1,k]\right)\\
&\quad - \frac{1}{2}\left(\sum_{k=1}^{n+1} \frac{(-1)^{m-1}}{2^{n+m-k}}\tbinom{n+m-(k+1)}{m-2} f[-1,t;k,1] + \sum_{k=1}^{m-1} \frac{(-1)^{m-1-k}}{2^{n+m-k}}\tbinom{n+m-(k+1)}{n} f[t,1;1,k]\right)\\
&= \frac{1}{2}\left(\sum_{k=1}^{n} \frac{(-1)^m}{2^{n+m-k}}\left(\tbinom{n+m-(k+1)}{m-1} + \tbinom{n+m-(k+1)}{m-2}\right) f[-1,t;k,1]\right)\\
&\quad + \frac{1}{2}\left(\sum_{k=1}^{m-1} \frac{(-1)^{m-k}}{2^{n+m-k}}\left(\tbinom{n+m-(k+1)}{n-1} + \tbinom{n+m-(k+1)}{n}\right) f[t,1;1,k]\right)\\
&\quad + \frac{1}{2}\frac{(-1)^{m-m}}{2^{n+m-m}}\tbinom{n+m-(m+1)}{n-1} f[t,1;1,m] - \frac{1}{2}\frac{(-1)^{m-1}}{2^{n+m-(n+1)}}\tbinom{n+m-(n+1+1)}{m-2} f[-1,t;n+1,1]\\
&= \sum_{k=1}^{n+1} \frac{(-1)^m}{2^{n+1+m-k}}\tbinom{n+1+m-(k+1)}{m-1} f[-1,t;k,1] + \sum_{k=1}^{m} \frac{(-1)^{m-k}}{2^{n+1+m-k}}\tbinom{n+1+m-(k+1)}{n} f[t,1;1,k],
\end{aligned}$$

which is equal to the right hand formula of $(*)$. In an analogous way to the above, we show that $(*)$ hold for the cases that $N + M \leq m + n + 1$.

Hence, we have shown the validity of $(*)$. Furthermore, since it holds that

$$\binom{N+M}{M} = \binom{N+M}{N},$$

$$\binom{N+M}{M}\left(\frac{N}{N+M}\right)^k \geq \binom{N+M-(k+1)}{M-1} \text{ for } k = 1,\dots,N$$

and

$$\binom{N+M}{N}\left(\frac{M}{N+M}\right)^k \geq \binom{N+M-(k+1)}{N-1} \text{ for } k = 1,\dots,M,$$

we have

$$\begin{aligned}
&|f[-1,t,1;N,1,M]|\\
&\leq \frac{1}{2^{N+M}}\sum_{k=1}^{N} 2^k \tbinom{N+M-(k+1)}{M-1} |f[-1,t;k,1]| + \frac{1}{2^{N+M}}\sum_{k=1}^{M} 2^k \tbinom{N+M-(k+1)}{N-1} |f[t,1;1,k]|\\
&\leq \frac{1}{2^{N+M}}\binom{N+M}{M}\left(\sum_{k=1}^{N}\left(\frac{2N}{N+M}\right)^k |f[-1,t;k,1]| + \sum_{k=1}^{M}\left(\frac{2M}{N+M}\right)^k |f[t,1;1,k]|\right).
\end{aligned}$$

□

Proposition 2.2.6. *Let m, n be positive integers. Let δ_1 be a real number with $\delta_1 > \frac{n-m}{n+m} - (-1)$ and δ_2 a real number with $\delta_2 > 1 - \frac{n-m}{n+m}$, where $\frac{n-m}{n+m}$ is the point which divides the interval $[-1, 1]$ in the ratio $n : m$. Let f be a piecewise analytic function*

$$f(x) = \begin{cases} p(x) & x \in \left[\frac{n-m}{n+m}, \infty\right) \\ q(x) & x \in \left(-\infty, \frac{n-m}{n+m}\right) \end{cases}$$

such that f is equal to an analytic function p on $\left[\frac{n-m}{n+m}, \infty\right)$ which has the Taylor expansion of p about 1 on $(1-\delta_2, 1+\delta_2)$, and f is equal to an analytic function q on $\left(-\infty, \frac{n-m}{n+m}\right)$ which has the Taylor expansion of q about -1 on $(-1-\delta_1, -1+\delta_1)$. Let α be the real number with $\alpha < -1$ and $|(\alpha+1)^n(\alpha-1)^m| = \frac{2^{n+m}\cdot n^n \cdot m^m}{(n+m)^{n+m}}$ and β the real number with $\beta > 1$ and $|(\beta+1)^n(\beta-1)^m| = \frac{2^{n+m}\cdot n^n\cdot m^m}{(n+m)^{n+m}}$. Then, the following hold:
(i) *There exists an $N \in \mathbf{N}$ such that for each $t \in [\alpha, \beta] \setminus \left\{\frac{n-m}{n+m}\right\}$, there exist real constants C_1, C_2, r_1 $\left(0 < r_1 < \frac{n+m}{2n}\right)$, r_2 $\left(0 < r_2 < \frac{n+m}{2m}\right)$ such that*

$$|f[-1, t; i, 1]| \le C_1 r_1^i, \ \ i \ge N,$$

and

$$|f[t, 1; 1, i]| \le C_2 r_2^i, \ \ i \ge N.$$

(ii) *If $p\left(\frac{n-m}{n+m}\right) = q\left(\frac{n-m}{n+m}\right)$, there exists an $N \in \mathbf{N}$ such that for each $t \in [\alpha, \beta]$, there exist real constants C_1, C_2, r_1 $(0 < r_1 < \frac{n+m}{2n})$, r_2 $\left(0 < r_2 < \frac{n+m}{2m}\right)$ such that*

$$|f[-1, t; i, 1]| \le C_1 r_1^i, \ \ i \ge N,$$

and

$$|f[t, 1; 1, i]| \le C_2 r_2^i, \ \ i \ge N.$$

Proof. Since the proof of (ii) can be reduced to that of (i), we prove (i). And we only show $|f[-1, t; i, 1]| \le C_1 r_1^i,\ i \in \mathbf{N}$ because $|f[t, 1; 1, i]| \le C_2 r_2^i,\ i \in \mathbf{N}$ are analogously shown. Let R_1, R_2 be real numbers with $\delta_1 > R_1 > \frac{2n}{n+m}$ and $\delta_2 > R_2 > \frac{2m}{n+m}$. From the assumption, q has the Taylor expansion of q about -1 on $[-1-R_1, -1+R_1]$,

$$q(x) = \sum_{j=0}^{\infty} \frac{q^{(j)}(-1)}{j!}(x+1)^j, \ x \in [-1-R_1, -1+R_1].$$

Hence, there exists a positive integer N_1 such that

$$\left| \frac{q^{(j)}(-1)}{j!} R_1^j \right| < 1, \ \ j \ge N_1.$$

And we have

$$\frac{|q^{(j)}(-1)|}{j!} < \frac{1}{R_1^j}, \ \ j \ge N_1. \qquad (**)$$

Now, we consider estimations of $|f[-1, t; i, 1]|$ for the cases that (1) $t \in \left[\alpha, \frac{n-m}{n+m}\right)$ and (2) $t \in \left(\frac{n-m}{n+m}, \beta\right]$.

Case (1). Since $f(t) = q(t), t \in \left[\alpha, \frac{n-m}{n+m}\right)$, by using Proposition 2.2.3 for $t \neq -1$, we obtain

$$\begin{aligned}
f[-1,t;i,1] &= \frac{1}{(t+1)^i}\left(f(t) - \sum_{j=0}^{i-1}\frac{q^{(j)}(-1)}{j!}(t+1)^j\right) \\
&= \frac{1}{(t+1)^i}\left(q(t) - \sum_{j=0}^{i-1}\frac{q^{(j)}(-1)}{j!}(t+1)^j\right) \\
&= \frac{1}{(t+1)^i}\sum_{j=i}^{\infty}\frac{q^{(j)}(-1)}{j!}(t+1)^j = \sum_{j=0}^{\infty}\frac{q^{(i+j)}(-1)}{(i+j)!}(t+1)^j.
\end{aligned}$$

For $t = -1$, since

$$f[-1,t;i,1] = f[-1;i+1] = \frac{q^{(i)}(-1)}{i!},$$

the equality stated above also holds. Noting that $R_1 > \max\left\{-1-\alpha, \frac{n-m}{n+m} - (-1)\right\}$, $|t+1| < R_1$ and from $(**)$, for each positive integer i with $i \geq N_1$, we have

$$\begin{aligned}
|f[-1,t;i,1]| &\leq \sum_{j=0}^{\infty}\left|\frac{q^{(i+j)}(-1)}{(i+j)!}\right||t+1|^j \\
&\leq \left(\frac{1}{R_1}\right)^i\sum_{j=0}^{\infty}\left(\frac{|t+1|}{R_1}\right)^j < \frac{1}{1 - \frac{\frac{2n}{n+m}}{R_1}}\left(\frac{1}{R_1}\right)^i.
\end{aligned}$$

From the definition of R_1, it follows that $0 < \frac{1}{R_1} < \frac{n+m}{2n}$.
Case (2). Since $f(t) = p(t), t \in \left(\frac{n-m}{n+m}, \beta\right]$, by using Proposition 2.2.3 we have

$$f[-1,t;i,1] = \frac{1}{(t+1)^i}\left(p(t) - \sum_{j=0}^{i-1}\frac{q^{(j)}(-1)}{j!}(t+1)^j\right).$$

Since p is continuous on $[1-R_2, 1+R_2]\left(\supset\left(\frac{n-m}{n+m}, \beta\right]\right)$, putting

$$M_1 = \max_{x\in[1-R_2,1+R_2]}|p(x)|,$$

we have

$$\begin{aligned}
|f[-1,t;i,1]| &\leq \frac{|p(t)|}{(t+1)^i} + \frac{1}{(t+1)^i}\left|\sum_{j=0}^{i-1}\frac{q^{(j)}(-1)}{j!}(t+1)^j\right| \\
&\leq M_1\cdot\left(\frac{1}{t+1}\right)^i + \frac{1}{(t+1)^i}\left|\sum_{j=0}^{i-1}\frac{q^{(j)}(-1)}{j!}(t+1)^j\right|.
\end{aligned}$$

To estimate $\frac{1}{(t+1)^i}\left|\sum_{j=0}^{i-1}\frac{q^{(j)}(-1)}{j!}(t+1)^j\right|$, we consider the cases that

$$\text{(a) } t \in \left(\frac{n-m}{n+m}, -1+R_1\right]$$

and

$$\text{(b) } t \in (-1+R_1, \beta].$$

Case (2-a). Since q has the Taylor expansion of q about -1 on $(-1-\delta_1, -1+\delta_1)$ and the sequence of functions $\left\{\sum_{j=0}^{N} \frac{q^{(j)}(-1)}{j!}(t+1)^j\right\}_{N\geq 0}$ is uniformly bounded on $[-1-R_1, -1+R_1]$, there exists a positive number M_2 such that

$$\left|\sum_{j=0}^{N} \frac{q^{(j)}(-1)}{j!}(t+1)^j\right| < M_2,\ N \in \{0,1,2,\cdots\},\ t \in [-1-R_1, -1+R_1].$$

Easily seeing that

$$\frac{1}{(t+1)^i}\left|\sum_{j=0}^{i-1} \frac{q^{(j)}(-1)}{j!}(t+1)^j\right| \leq M_2 \cdot \left(\frac{1}{t+1}\right)^i,$$

we get

$$|f[-1,t;i,1]| \leq (M_1+M_2)\cdot\left(\frac{1}{t+1}\right)^i.$$

Since $t+1 \in \left(\frac{2n}{n+m}, R_1\right]$, $0 < \frac{1}{t+1} < \frac{n+m}{2n}$ hold.
(2-b) For each positive integer i with $i \geq N_1+1$, noticing that $t+1 \in (R_1, \beta+1]$, we have

$$\begin{aligned}
&\frac{1}{(t+1)^i}\left|\sum_{j=0}^{i-1} \frac{q^{(j)}(-1)}{j!}(t+1)^j\right| \\
&\leq \frac{1}{(t+1)^i}\left|\sum_{j=0}^{N_1-1} \frac{q^{(j)}(-1)}{j!}(t+1)^j\right| + \frac{1}{(t+1)^i}\sum_{j=N_1}^{i-1}\left(\frac{t+1}{R_1}\right)^j \\
&\leq \sum_{j=0}^{N_1-1} \frac{|q^{(j)}(-1)|}{j!}(\beta+1)^j \cdot \left(\frac{1}{R_1}\right)^i + \frac{1}{\dfrac{t+1}{R_1}-1}\cdot\left(\frac{1}{R_1}\right)^i.
\end{aligned}$$

Therefore, we get

$$|f[-1,t;i,1]| \leq \left(M_1 + \sum_{j=0}^{N_1-1} \frac{|q^{(j)}(-1)|}{j!}(\beta+1)^j + \frac{1}{\dfrac{t+1}{R_1}-1}\right)\left(\frac{1}{R_1}\right)^i.$$

As is seen in the case (1), $\frac{1}{R_1}$ satisfies $0 < \frac{1}{R_1} < \frac{n+m}{2n}$.

Consequently, for each $t \in [\alpha, \beta] \setminus \left\{\frac{n-m}{n+m}\right\}$, there exist C_1 and r_1 $\left(0 < r_1 < \frac{n+m}{2n}\right)$ such that

$$|f[-1,t;i,1]| \leq C_1 r_1^i,\ i \geq N_1+1,$$

which leads to the validity of (i).

$\square$

Corollary 2.2.7. *Let m, n be positive integers. Let δ_1 be a real number with $\delta_1 > \frac{n-m}{n+m} - (-1)$ and δ_2 a real number with $\delta_2 > 1 - \frac{n-m}{n+m}$, where $\frac{n-m}{n+m}$ is the point which divides the interval $[-1, 1]$ in the ratio $n : m$. Let R_1, R_2 be real numbers with $\delta_1 > R_1 > \frac{2n}{n+m}$ and $\delta_2 > R_2 > \frac{2m}{n+m}$. Let f be a piecewise analytic function*

$$f(x) = \begin{cases} p(x) & x \in \left[\frac{n-m}{n+m}, \infty\right) \\ q(x) & x \in \left(-\infty, \frac{n-m}{n+m}\right) \end{cases}$$

such that f is equal to an analytic function p on $\left[\frac{n-m}{n+m}, \infty\right)$ which has the Taylor expansion of p about 1 on $(1 - \delta_2, 1 + \delta_2)$, and f is equal to an analytic function q on $\left(-\infty, \frac{n-m}{n+m}\right)$ which has the Taylor expansion of q about -1 on $(-1 - \delta_1, -1 + \delta_1)$. Let α be the real number with $\alpha < -1$ and $|(\alpha+1)^n(\alpha-1)^m| = \frac{2^{n+m} \cdot n^n \cdot m^m}{(n+m)^{n+m}}$ and β the real number with $\beta > 1$ and $|(\beta+1)^n(\beta-1)^m| = \frac{2^{n+m} \cdot n^n \cdot m^m}{(n+m)^{n+m}}$. Let the functions $C_1(t)$, $r_1(t)$, $C_2(t)$ and $r_2(t)$ on $[\alpha, \beta]$ be defined as follows:

$$C_1(t) = \begin{cases} 1 & , \ t \in [\alpha, -1 + R_1] \\ 1 + \dfrac{1}{\dfrac{t+1}{R_1} - 1} & , \ t \in (-1 + R_1, \beta] \end{cases} ,$$

$$r_1(t) = \begin{cases} \dfrac{1}{R_1} & , \ t \in \left[\alpha, \dfrac{n-m}{n+m}\right] \\ \dfrac{1}{t+1} & , \ t \in \left(\dfrac{n-m}{n+m}, -1 + R_1\right] \\ \dfrac{1}{R_1} & , \ t \in (-1 + R_1, \beta] \end{cases} ,$$

$$C_2(t) = \begin{cases} 1 & , \ t \in [1 - R_2, \beta] \\ 1 + \dfrac{1}{\dfrac{1-t}{R_2} - 1} & , \ t \in [\alpha, 1 - R_2) \end{cases} ,$$

$$r_2(t) = \begin{cases} \dfrac{1}{R_2} & , \ t \in \left[\dfrac{n-m}{n+m}, \beta\right] \\ \dfrac{1}{1-t} & , \ t \in \left[1 - R_2, \dfrac{n-m}{n+m}\right) \\ \dfrac{1}{R_2} & , \ t \in [\alpha, 1 - R_2) \end{cases} .$$

Then, the following hold:
(i) *There exist $C > 0$, $N \in \mathbf{N}$ such that for each $t \in [\alpha, \beta] \setminus \left\{\frac{n-m}{n+m}\right\}$,*

$$|f[-1, t; i, 1]| \leq CC_1(t)(r_1(t))^i, \ i \geq N,$$

and

$$|f[t, 1; 1, i]| \leq CC_2(t)(r_2(t))^i, \ i \geq N.$$

(ii) *If* $p\left(\frac{n-m}{n+m}\right) = q\left(\frac{n-m}{n+m}\right)$, *there exist* $C > 0$, $N \in \mathbf{N}$ *such that for each* $t \in [\alpha, \beta]$,

$$|f[-1, t; i, 1]| \leq CC_1(t)(r_1(t))^i, \ i \geq N,$$

and

$$|f[t, 1; 1, i]| \leq CC_2(t)(r_2(t))^i, \ i \geq N.$$

Proposition 2.2.8. *Let* m, n, N *be positive integers. Let* δ_1 *be a real number with* $\delta_1 > \frac{n-m}{n+m} - (-1)$ *and* δ_2 *a real number with* $\delta_2 > 1 - \frac{n-m}{n+m}$, *where* $\frac{n-m}{n+m}$ *is the point which divides the interval* $[-1, 1]$ *in the ratio* $n : m$. *Let* f *be a piecewise analytic function*

$$f(x) = \begin{cases} p(x) & x \in \left[\frac{n-m}{n+m}, \infty\right) \\ q(x) & x \in \left(-\infty, \frac{n-m}{n+m}\right) \end{cases}$$

such that f *is equal to an analytic function* p *on* $\left[\frac{n-m}{n+m}, \infty\right)$ *which has the Taylor expansion of* p *about* 1 *on* $(1 - \delta_2, 1 + \delta_2)$, *and* f *is equal to an analytic function* q *on* $\left(-\infty, \frac{n-m}{n+m}\right)$ *which has the Taylor expansion of* q *about* -1 *on* $(-1 - \delta_1, -1 + \delta_1)$. *Let* α *be the real number with* $\alpha < -1$ *and* $|(\alpha + 1)^n(\alpha - 1)^m| = \frac{2^{n+m} \cdot n^n \cdot m^m}{(n+m)^{n+m}}$ *and* β *the real number with* $\beta > 1$ *and* $|(\beta + 1)^n(\beta - 1)^m| = \frac{2^{n+m} \cdot n^n \cdot m^m}{(n+m)^{n+m}}$. *Then, there exist numbers* $M_1, M_2 \in \mathbf{R}$ *such that*

$$|f[-1, t; i, 1]| \leq M_1$$

and

$$|f[t, 1; 1, i]| \leq M_2$$

for each $i = 1, 2, \ldots, N$ *and for each* $t \in [\alpha, \beta]$.

Proof. We only prove $|f[-1, t; i, 1]| \leq M_1$. Let us recall that from Taylor's theorem, for any $t \in [\alpha, \frac{n-m}{n+m})$ there exists an $a \in \left[\alpha, \frac{n-m}{n+m}\right]$ such that

$$\frac{1}{(t+1)^i}\left(q(t) - \sum_{j=0}^{i-1} \frac{q^{(j)}(-1)}{j!}(t+1)^j\right) = \frac{1}{(t+1)^i}\frac{q^{(i)}(a)}{i!}(t+1)^i = \frac{q^{(i)}(a)}{i!}.$$

Therefore, we have

$$\begin{aligned} &|f[-1, t; i, 1]| \\ &= \begin{cases} \left|\dfrac{1}{(t+1)^i}\left(p(t) - \displaystyle\sum_{j=0}^{i-1} \frac{q^{(j)}(-1)}{j!}(t+1)^j\right)\right| & , \ t \in \left[\dfrac{n-m}{n+m}, \beta\right] \\ \left|\dfrac{1}{(t+1)^i}\left(q(t) - \displaystyle\sum_{j=0}^{i-1} \frac{q^{(j)}(-1)}{j!}(t+1)^j\right)\right| & , \ t \in \left[\alpha, \dfrac{n-m}{n+m}\right) \end{cases} \\ &\leq \begin{cases} \dfrac{1}{\left(\frac{n-m}{n+m} + 1\right)^i}\left(\displaystyle\max_{x \in \left[\frac{n-m}{n+m}, \beta\right]} |p(x)| + \sum_{j=0}^{i-1} \frac{|q^{(j)}(-1)|}{j!}(\beta+1)^j\right) & , \ t \in \left[\dfrac{n-m}{n+m}, \beta\right] \\ \displaystyle\max_{x \in \left[\alpha, \frac{n-m}{n+m}\right]} \frac{|q^{(i)}(x)|}{i!} & , \ t \in \left[\alpha, \dfrac{n-m}{n+m}\right) \end{cases}. \end{aligned}$$

Putting

$$M_1 = \max_{i=1,\dots,N} C_i,$$

where

$$C_i = \max\left\{ \frac{1}{\left(\frac{n-m}{n+m}+1\right)^i} \left(\max_{x\in\left[\frac{n-m}{n+m},\beta\right]} |p(x)| + \sum_{j=0}^{i-1} \frac{|q^{(j)}(-1)|}{j!}(\beta+1)^j \right), \max_{x\in\left[\alpha,\frac{n-m}{n+m}\right]} \frac{|q^{(i)}(x)|}{i!} \right\},$$

we obtain for each $i = 1, \dots, N$,

$$|f[-1,t;i,1]| \leq M_1, \; t \in [\alpha,\beta].$$

$\square$

Proposition 2.2.9. *Let m, n be positive integers. Let δ_1 be a real number with $\delta_1 > \frac{n-m}{n+m} - (-1)$ and δ_2 a real number with $\delta_2 > 1 - \frac{n-m}{n+m}$, where $\frac{n-m}{n+m}$ is the point which divides the interval $[-1,1]$ in the ratio $n : m$. Let R_1, R_2 be real numbers with $\delta_1 > R_1 > \frac{2n}{n+m}$ and $\delta_2 > R_2 > \frac{2m}{n+m}$. Let f be a piecewise analytic function*

$$f(x) = \begin{cases} p(x) & x \in \left[\frac{n-m}{n+m}, \infty\right) \\ q(x) & x \in \left(-\infty, \frac{n-m}{n+m}\right) \end{cases}$$

such that f is equal to an analytic function p on $\left[\frac{n-m}{n+m}, \infty\right)$ which has the Taylor expansion of p about 1 on $(1-\delta_2, 1+\delta_2)$, and f is equal to an analytic function q on $\left(-\infty, \frac{n-m}{n+m}\right)$ which has the Taylor expansion of q about -1 on $(-1-\delta_1, -1+\delta_1)$. Let α be the real number with $\alpha < -1$ and $|(\alpha+1)^n(\alpha-1)^m| = \frac{2^{n+m}\cdot n^n\cdot m^m}{(n+m)^{n+m}}$ and β the real number with $\beta > 1$ and $|(\beta+1)^n(\beta-1)^m| = \frac{2^{n+m}\cdot n^n\cdot m^m}{(n+m)^{n+m}}$. Let the functions $C_1(t)$, $r_1(t)$, $C_2(t)$ and $r_2(t)$ on $[\alpha,\beta]$ be defined as follows:

$$C_1(t) = \begin{cases} 1 & , \; t \in [\alpha, -1+R_1] \\ 1 + \dfrac{1}{\dfrac{t+1}{R_1} - 1} & , \; t \in (-1+R_1, \beta] \end{cases},$$

$$r_1(t) = \begin{cases} \dfrac{1}{R_1} & , \; t \in \left[\alpha, \dfrac{n-m}{n+m}\right] \\ \dfrac{1}{t+1} & , \; t \in \left(\dfrac{n-m}{n+m}, -1+R_1\right] \\ \dfrac{1}{R_1} & , \; t \in (-1+R_1, \beta] \end{cases},$$

$$C_2(t) = \begin{cases} 1 & , \ t \in [1-R_2, \beta] \\ 1 + \dfrac{1}{\dfrac{1-t}{R_2} - 1} & , \ t \in [\alpha, 1-R_2) \end{cases} ,$$

$$r_2(t) = \begin{cases} \dfrac{1}{R_2} & , \ t \in \left[\dfrac{n-m}{n+m}, \beta\right] \\ \dfrac{1}{1-t} & , \ t \in \left[1-R_2, \dfrac{n-m}{n+m}\right) \\ \dfrac{1}{R_2} & , \ t \in [\alpha, 1-R_2) \end{cases} .$$

Then, the following hold:
(i) *For each* $t \in [\alpha, \beta]$*,*

$$0 < r_1(t) < \frac{n+m}{2n},$$

and

$$0 < r_2(t) < \frac{n+m}{2m}.$$

(ii) *There exists a* $C > 0$ *such that for each* $t \in [\alpha, \beta] \setminus \left\{\frac{n-m}{n+m}\right\}$*,*

$$|f[-1, t; i, 1]| \le CC_1(t)(r_1(t))^i, \ \ i \in \mathbf{N},$$

and

$$|f[t, 1; 1, i]| \le CC_2(t)(r_2(t))^i, \ \ i \in \mathbf{N}.$$

(iii) *If* $p\left(\frac{n-m}{n+m}\right) = q\left(\frac{n-m}{n+m}\right)$*, there exists a* $C > 0$ *such that for each* $t \in [\alpha, \beta]$*,*

$$|f[-1, t; i, 1]| \le CC_1(t)(r_1(t))^i, \ \ i \in \mathbf{N},$$

and

$$|f[t, 1; 1, i]| \le CC_2(t)(r_2(t))^i, \ \ i \in \mathbf{N}.$$

Proof. (i) can be easily obtained from the definition of $r_1(t), r_2(t)$. We only prove (ii) since we can prove (iii) similarly to (ii).

From Corollary 2.2.7, there exist $C_0 > 0$, $N \in \mathbf{N}$ such that for each $t \in [\alpha, \beta] \setminus \left\{\frac{n-m}{n+m}\right\}$,

$$|f[-1, t; i, 1]| \le C_0 C_1(t)(r_1(t))^i, \ \ i \ge N,$$

and

$$|f[t, 1; 1, i]| \le C_0 C_2(t)(r_2(t))^i, \ \ i \ge N.$$

Also, from Proposition 2.2.8, there exists $M \in \mathbf{R}$ such that

$$|f[-1, t; i, 1]| \le M,$$

and

$$|f[t, 1; 1, i]| \le M$$

for each $i = 1, 2, \dots, N-1$ and for each $t \in [\alpha, \beta]$. We put

$$C = \max\left\{C_0, R_1 M, \dots, R_1^{N-1} M, R_2 M, \dots, R_2^{N-1} M\right\}.$$

Now, we prove

$$|f[-1, t; i, 1]| \le C C_1(t) (r_1(t))^i$$

for each $t \in [\alpha, \beta] \setminus \left\{\frac{n-m}{n+m}\right\}$ and for each $i \in \mathbf{N}$ by considering the cases that (1) $t \in \left[\alpha, \frac{n-m}{n+m}\right)$, (2) $t \in \left(\frac{n-m}{n+m}, -1 + R_1\right]$ and (3) $t \in (-1 + R_1, \beta]$.
Case (1). We have for each $i \ge N$,

$$|f[-1, t; i, 1]| \le C_0 \cdot \left(\frac{1}{R_1}\right)^i \le C \cdot \left(\frac{1}{R_1}\right)^i.$$

Also, we obtain for each $i = 1, \dots, N-1$,

$$|f[-1, t; i, 1]| \le M = M R_1^i \cdot \left(\frac{1}{R_1}\right)^i \le C \cdot \left(\frac{1}{R_1}\right)^i.$$

Case (2). We have for each $i \ge N$,

$$|f[-1, t; i, 1]| \le C_0 \cdot \left(\frac{1}{t+1}\right)^i \le C \cdot \left(\frac{1}{t+1}\right)^i.$$

Also, we obtain for each $i = 1, \dots, N-1$,

$$\begin{aligned} |f[-1, t; i, 1]| &\le M \\ &= M (t+1)^i \cdot \left(\frac{1}{t+1}\right)^i \\ &\le M R_1^i \cdot \left(\frac{1}{t+1}\right)^i \\ &\le C \cdot \left(\frac{1}{t+1}\right)^i. \end{aligned}$$

Case (3). We have for each $i \ge N$,

$$|f[-1, t; i, 1]| \le C_0 \cdot \left(1 + \frac{1}{\frac{t+1}{R_1} - 1}\right) \cdot \left(\frac{1}{R_1}\right)^i \le C \cdot \left(1 + \frac{1}{\frac{t+1}{R_1} - 1}\right) \cdot \left(\frac{1}{R_1}\right)^i.$$

Also, we obtain for each $i = 1, \ldots, N-1$,

$$
\begin{aligned}
|f[-1,t;i,1]| &\leq M \\
&= M \cdot \frac{1}{\left(1+\dfrac{1}{\dfrac{t+1}{R_1}-1}\right)\cdot\left(\dfrac{1}{R_1}\right)^i} \cdot \left(1+\dfrac{1}{\dfrac{t+1}{R_1}-1}\right)\cdot\left(\dfrac{1}{R_1}\right)^i \\
&= MR_1^i \cdot \frac{t+1-R_1}{t+1} \cdot \left(1+\dfrac{1}{\dfrac{t+1}{R_1}-1}\right)\cdot\left(\dfrac{1}{R_1}\right)^i \\
&\leq MR_1^i \cdot \left(1+\dfrac{1}{\dfrac{t+1}{R_1}-1}\right)\cdot\left(\dfrac{1}{R_1}\right)^i \\
&\leq C\left(1+\dfrac{1}{\dfrac{t+1}{R_1}-1}\right)\cdot\left(\dfrac{1}{R_1}\right)^i.
\end{aligned}
$$

Similarly, we have

$$|f[t,1;1,i]| \leq CC_2(t)(r_2(t))^i$$

for each $t \in [\alpha,\beta] \setminus \left\{\frac{n-m}{n+m}\right\}$ and for each $i \in \mathbf{N}$.

□

2.3 Proof of Theorem 2.1.1

Now we are in position to prove Theorem 2.1.1.

Proof of Theorem 2.1.1. (1) Since we easily see that

$$|(t+1)^n(t-1)^m| \leq \frac{2^{n+m}\cdot n^n \cdot m^m}{(n+m)^{n+m}},\ t \in [\alpha,\beta],$$

from Proposition 2.2.3, for each $t \in [\alpha,\beta]$, we have

$$
\begin{aligned}
|f(t) - p_{f,\{-1,1\}(n\ell,m\ell)}(t)| &= |f[-1,t,1;n\ell,1,m\ell]| \cdot |(t+1)^n(t-1)^m|^\ell \\
&\leq |f[-1,t,1;n\ell,1,m\ell]| \cdot \left(\frac{2^{n+m}\cdot n^n\cdot m^m}{(n+m)^{n+m}}\right)^\ell.
\end{aligned}
$$

On the other hand, by using Proposition 2.2.5, Proposition 2.2.9 and Stirling's formula, there exist positive numbers C_0, C_3 satisfying that for each $t \in [\alpha, \beta] \setminus \left\{\frac{n-m}{n+m}\right\}$

$$
\begin{aligned}
&|f[-1,t,1;n\ell,1,m\ell]| \\
&\le \frac{1}{2^{(n+m)\ell}}\binom{(n+m)\ell}{m\ell}\left(\sum_{k=1}^{n\ell}\left(\frac{2n}{n+m}\right)^k |f[-1,t;k,1]| + \sum_{k=1}^{m\ell}\left(\frac{2m}{n+m}\right)^k |f[t,1;1,k]|\right) \\
&\le \frac{C_0}{2^{(n+m)\ell}}\binom{(n+m)\ell}{m\ell}\left(\sum_{k=1}^{n\ell}\left(\frac{2n}{n+m}\right)^k C_1(t)\big(r_1(t)\big)^k + \sum_{k=1}^{m\ell}\left(\frac{2m}{n+m}\right)^k C_2(t)\big(r_1(t)\big)^k\right) \\
&\le \frac{C_0}{2^{(n+m)\ell}}\binom{(n+m)\ell}{m\ell}\left(\frac{C_1(t)}{1-\dfrac{2n}{n+m}\cdot r_1(t)} + \frac{C_2(t)}{1-\dfrac{2m}{n+m}\cdot r_2(t)}\right) \\
&\le \frac{C_0C_3}{2^{(n+m)\ell}}\frac{1}{\sqrt{\ell}}\left(\frac{(n+m)^{n+m}}{n^n\cdot m^m}\right)^\ell\left(\frac{C_1(t)}{1-\dfrac{2n}{n+m}\cdot r_1(t)} + \frac{C_2(t)}{1-\dfrac{2m}{n+m}\cdot r_2(t)}\right) \\
&= C_0C_3\left(\frac{C_1(t)}{1-\dfrac{2n}{n+m}\cdot r_1(t)} + \frac{C_2(t)}{1-\dfrac{2m}{n+m}\cdot r_2(t)}\right)\frac{1}{\sqrt{\ell}}\left(\frac{(n+m)^{n+m}}{2^{n+m}\cdot n^n\cdot m^m}\right)^\ell.
\end{aligned}
$$

Putting

$$C(t) = C_0C_3\left(\frac{C_1(t)}{1-\dfrac{2n}{n+m}\cdot r_1(t)} + \frac{C_2(t)}{1-\dfrac{2m}{n+m}\cdot r_2(t)}\right),$$

we obtain for each $t \in [\alpha, \beta] \setminus \left\{\frac{n-m}{n+m}\right\}$,

$$|f(t) - p_{f,\{-1,1\}(n\ell,m\ell)}(t)| \le \frac{C(t)}{\sqrt{\ell}}\left(\frac{(n+m)^{n+m}}{2^{n+m}\cdot n^n\cdot m^m}\right)^\ell\cdot\left(\frac{2^{n+m}\cdot n^n\cdot m^m}{(n+m)^{n+m}}\right)^\ell = \frac{C(t)}{\sqrt{\ell}}.$$

We can prove (3) in a similar way to the proof of (1).

(2) We show $C(t)$ is bounded on $[\alpha, a] \cup [b, \beta]$ by proving the following functions are bounded on $[\alpha, a] \cup [b, \beta]$.

$$\text{(i) } C_1(t) \quad \text{(ii) } \frac{1}{1-\dfrac{2n}{n+m}\cdot r_1(t)} \quad \text{(iii) } C_2(t) \quad \text{(iv) } \frac{1}{1-\dfrac{2m}{n+m}\cdot r_2(t)}$$

From Proposition 2.2.9, the functions C_1, C_2, r_1, r_2 are expressed as follows:

$$C_1(t) = \begin{cases} 1 & , \ t \in [\alpha, -1+R_1] \\ 1 + \dfrac{1}{\dfrac{t+1}{R_1} - 1} & , \ t \in (-1+R_1, \beta] \end{cases} ,$$

$$r_1(t) = \begin{cases} \dfrac{1}{R_1} & , \ t \in \left[\alpha, \dfrac{n-m}{n+m}\right] \\ \dfrac{1}{t+1} & , \ t \in \left(\dfrac{n-m}{n+m}, -1+R_1\right] \\ \dfrac{1}{R_1} & , \ t \in (-1+R_1, \beta] \end{cases} ,$$

$$C_2(t) = \begin{cases} 1 & , \ t \in [1-R_2, \beta] \\ 1 + \dfrac{1}{\dfrac{1-t}{R_2} - 1} & , \ t \in [\alpha, 1-R_2) \end{cases} ,$$

$$r_2(t) = \begin{cases} \dfrac{1}{R_2} & , \ t \in \left[\dfrac{n-m}{n+m}, \beta\right] \\ \dfrac{1}{1-t} & , \ t \in \left[1-R_2, \dfrac{n-m}{n+m}\right) \\ \dfrac{1}{R_2} & , \ t \in [\alpha, 1-R_2) \end{cases} .$$

Therefore, let a_1, a_2 be the real numbers with

$$0 < a_1 < \min\left\{b - \frac{n-m}{n+m}, \delta_1 - \frac{2n}{n+m}\right\},$$

$$0 < a_2 < \min\left\{\frac{n-m}{n+m} - a, \delta_2 - \frac{2m}{n+m}\right\},$$

by putting $R_1 = \frac{2n}{n+m} + a_1$, $R_2 = \frac{2m}{n+m} + a_2$, we can see that functions (i), (ii), (iii) and (iv) are bounded on $[\alpha, a] \cup [b, \beta]$.

$\square$

Chapter 3

Two point Taylor expansion of Heaviside function

3.1 Main Result

The purpose of this chapter is to prove the following theorem.

Theorem 3.1.1. *Let m, n be positive integers. Let f be a real-valued function on $\mathbf{R}$ which is expressed as*

$$f(x) = \begin{cases} C_1 & x \in \left[\frac{n-m}{n+m}, \infty\right) \\ C_2 & x \in \left(-\infty, \frac{n-m}{n+m}\right) \end{cases},$$

where C_1 and C_2 are real numbers. Let $p_{f,\{-1,1\}(n\ell,m\ell)}, \ell \in \mathbf{N}$ be the Hermite interpolating polynomials to f at $-1, 1$ with multiplicities $n\ell, m\ell$. Then, there exists a positive number C such that

$$\left| p_{f,\{-1,1\}(n\ell,m\ell)}\left(\frac{n-m}{n+m}\right) - \frac{C_1+C_2}{2} \right| \leq \frac{C}{\sqrt{\ell}}, \quad \ell \in \mathbf{N}.$$

3.2 The normal approximation to the negative binomial distribution

To show Theorem 3.1.1, we need to prepare four propositions.

From Ex. 3 in Davis [4, p. 37], we obtain the following proposition.

Proposition 3.2.1. *Let a, b be distinct nodes and m, n positive integers. Let f be a sufficiently differentiable function at a, b. A, B are functions defined by*

$$A(x) = \frac{f(x)}{(x-b)^m}, \quad B(x) = \frac{f(x)}{(x-a)^n}.$$

Then, the polynomial $p_{f,\{a,b\}(n,m)}(x)$ is expressed as

$$p_{f,\{a,b\}(n,m)}(x) = (x-a)^n \sum_{k=0}^{m-1} \frac{B^{(k)}(b)}{k!}(x-b)^k + (x-b)^m \sum_{k=0}^{n-1} \frac{A^{(k)}(a)}{k!}(x-a)^k.$$

Proposition 3.2.2 (Durrett [5, p. 137])**.** *Let $X_1, X_2, \ldots$ be i.i.d with $EX_i = 0$, $EX_i^2 = \sigma^2$, and $E|X_i|^3 = \rho < \infty$. Then, for all $x \in \mathbf{R}$ and for all $N = 1, 2, \ldots$ it holds that*

$$\left| P\left(\frac{X_1 + \cdots + X_N}{\sigma\sqrt{N}} \le x \right) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy \right| \le \frac{3\rho}{\sigma^3} \frac{1}{\sqrt{N}}.$$

Proposition 3.2.3. *Let p be a real number with $0 < p < 1$. Let X be a geometric random variable with parameter p, that is,*

$$P(X = k) = p(1-p)^k, \ \ k = 0, 1, 2, \ldots.$$

Then, it holds that

$$E\left(|X - E(X)|^3\right) < \infty.$$

Proof. Since X is a geometric random variable with parameter p, the mean of X is

$$E(X) = \frac{1-p}{p},$$

and the variance of X is

$$V(X) = E(X^2) - (E(X))^2 = \frac{1-p}{p^2}.$$

Therefore, we get

$$\begin{aligned} E(X^2) &= (E(X))^2 + \frac{1-p}{p^2} \\ &= \left(\frac{1-p}{p}\right)^2 + \frac{1-p}{p^2} \\ &= \frac{2 - 3p + p^2}{p^2}. \end{aligned}$$

Now, we show that

$$E(X^3) = \frac{(1-p)(6 - 6p + p^2)}{p^3}.$$

Since we have

$$\begin{aligned} E\left((X+1)^3\right) &= \sum_{k=0}^{\infty} (k+1)^3 p(1-p)^k \\ &= \frac{1}{1-p} \sum_{k=0}^{\infty} (k+1)^3 p(1-p)^{k+1} \\ &= \frac{E(X^3)}{1-p}, \end{aligned}$$

we obtain

$$E\left((X+1)^3\right) - E(X^3) = \frac{p}{1-p} E(X^3).$$

Therefore, since we have

$$\begin{aligned} E\left((X+1)^3\right) - E(X^3) &= 3E(X^2) + 3E(X) + 1 \\ &= \frac{6-6p+p^2}{p^2}, \end{aligned}$$

we get

$$E(X^3) = \frac{(1-p)(6-6p+p^2)}{p^3}.$$

From the above, we obtain

$$\begin{aligned} & E(|X - E(X)|^3) \\ &= \sum_{k=0}^{\infty} \left| k - \frac{1-p}{p} \right|^3 p(1-p)^k \\ &\leq \sum_{k=0}^{\infty} k^3 p(1-p)^k + 3 \cdot \frac{1-p}{p} \sum_{k=0}^{\infty} k^2 p(1-p)^k + 3 \left(\frac{1-p}{p}\right)^2 \sum_{k=0}^{\infty} kp(1-p)^k \\ &\quad + \left(\frac{1-p}{p}\right)^3 \sum_{k=0}^{\infty} p(1-p)^k \\ &= E(X^3) + \frac{3(1-p)}{p} \cdot E(X^2) + 3\left(\frac{1-p}{p}\right)^2 E(X) + \left(\frac{1-p}{p}\right)^3 \\ &< \infty. \end{aligned}$$

$\square$

Proposition 3.2.4. *Let p be a real number with $0 < p < 1$. Then, there exists a positive number C such that*

$$\left| \sum_{\substack{k \in \mathbf{Z} \\ 0 \leq k \leq \left(\sqrt{\frac{1-p}{p^2}}\sqrt{N}\right)x + \frac{N(1-p)}{p}}} \binom{k+N-1}{k} p^N (1-p)^k - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy \right| \leq \frac{C}{\sqrt{N}}$$

for all $x \in \mathbf{R}$ and for all $N = 1, 2, \ldots$.

Proof. Let $X_1, X_2, \ldots$ be independent geometric random variables, where X_i has parameter p. Then, we have

$$E\left(X_i - E(X_i)\right) = 0,$$

$$V\left(X_i - E(X_i)\right) = V(X_i) = \frac{1-p}{p^2}.$$

From Proposition 3.2.3, we have $E\left(|X_i - E(X_i)|^3\right) < \infty$. Therefore, from Proposition 3.2.2 there exists a positive number C such that

$$\left| P\left(\frac{1}{\sqrt{\frac{1-p}{p^2}}\sqrt{N}} \sum_{k=1}^{N} \left(X_k - \frac{1-p}{p} \right) \leq x \right) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy \right| \leq \frac{C}{\sqrt{N}}$$

for all $x \in \mathbf{R}$ and for all $N = 1, 2, \ldots$. Since $\sum_{k=1}^{N} X_k$ obeys the negative binomial distribution $NB(N, p)$,

$$P\left(\sum_{k=1}^{N} X_k \leq x\right) = \sum_{k \in \{j \in \mathbf{Z} | 0 \leq j \leq x\}} \binom{k+N-1}{k} p^N (1-p)^k.$$

Hence, there exists a positive number C such that

$$\left| \sum_{\substack{k \in \mathbf{Z} \\ 0 \leq k \leq \left(\sqrt{\frac{1-p}{p^2}}\sqrt{N}\right)x + \frac{N(1-p)}{p}}} \binom{k+N-1}{k} p^N (1-p)^k - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy \right| \leq \frac{C}{\sqrt{N}}$$

for all $x \in \mathbf{R}$ and for all $N = 1, 2, \ldots$.

□

3.3 Proof of Theorem 3.1.1

In this section, we prove Theorem 3.1.1. Taguchi [20] already proved Proposition 1.3.9. Here we show a proof of Theorem 3.1.1 by Proposition 3.2.4 that the standard normal distribution can be approximated by negative binomial distributions.

Proof of Theorem 3.1.1. Without loss of generality, we can assume that

$$f(x) = \begin{cases} 1 & x \in \left[\frac{n-m}{n+m}, \infty\right) \\ 0 & x \in \left(-\infty, \frac{n-m}{n+m}\right) \end{cases}.$$

From Proposition 3.2.1, $p_{f,\{-1,1\}(n\ell,m\ell)}(x)$ is expressed as follows:

$$\begin{aligned} p_{f,\{-1,1\}(n\ell,m\ell)}(x) &= (x+1)^{n\ell} \sum_{k=0}^{m\ell-1} \frac{1}{k!} \left((z+1)^{-n\ell}\right)^{(k)}\Big|_{z=1} (x-1)^k \\ &= \sum_{k=0}^{m\ell-1} \binom{n\ell+k-1}{k} \left(\frac{x+1}{2}\right)^{n\ell} \left(\frac{1-x}{2}\right)^k. \end{aligned}$$

We get

$$p_{f,\{-1,1\}(n\ell,m\ell)}\left(\frac{n-m}{n+m}\right) = \sum_{k=0}^{m\ell-1} \binom{k+n\ell-1}{k} \left(\frac{n}{n+m}\right)^{n\ell} \left(\frac{m}{n+m}\right)^k.$$

Putting $N = n\ell,\ p = \frac{n}{n+m},\ x = -\sqrt{\frac{n}{(n+m)m\ell}}$, for each $\ell = 1, 2, \ldots$, we obtain

$$\left(\sqrt{\frac{1-p}{p^2}}\sqrt{N}\right)x + \frac{N(1-p)}{p} = m\ell - 1,$$

and

$$\left| p_{f,\{-1,1\}(n\ell,m\ell)}\left(\frac{n-m}{n+m}\right) - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\sqrt{\frac{n}{(n+m)m\ell}}} e^{-\frac{y^2}{2}}dy \right| \leq \frac{C}{\sqrt{n\ell}}.$$

Therefore, for each $\ell = 1, 2, \ldots$ we have

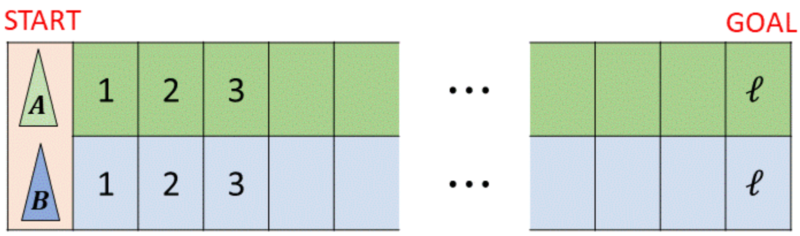
$$\begin{aligned}
&\left| p_{f,\{-1,1\}(n\ell,m\ell)}\left(\frac{n-m}{n+m}\right) - \frac{1}{2} \right| \\
&\leq \frac{C}{\sqrt{n\ell}} + \left| \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{-\frac{y^2}{2}}dy - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\sqrt{\frac{n}{(n+m)m\ell}}} e^{-\frac{y^2}{2}}dy \right| \\
&\leq \frac{C}{\sqrt{n\ell}} + \frac{1}{\sqrt{2\pi}}\sqrt{\frac{n}{(n+m)m\ell}} \\
&= \left(\frac{C}{\sqrt{n}} + \frac{1}{\sqrt{2\pi}}\sqrt{\frac{n}{(n+m)m}} \right)\frac{1}{\sqrt{\ell}}.
\end{aligned}$$

$\square$

Remark 3.3.1. (1) Let us show an intuitive interpretation of Theorem 1.3.5 by probability. Without loss of generality, we can assume that

$$f(x) = \begin{cases} 1 & x \in [0, \infty) \\ 0 & x \in (-\infty, 0) \end{cases}.$$

We consider the following game for two players A, B: A coin is tossed repeatedly. The probability of heads on any toss is $p = \frac{1}{2}$ and the probability of tails on any toss is $1 - p = \frac{1}{2}$. If the coin lands heads up, then player A goes forward 1 spaces. If the coin lands on the reverse, then player B goes forward 1 spaces. The distance from the start to player A's goal is ℓ spaces. The distance from the start to player B's goal is ℓ spaces. The player who reaches the first, wins.



Since $p_{f,\{-1,1\}(\ell,\ell)}$ is expressed as

$$p_{f,\{-1,1\}(\ell,\ell)}(x) = \sum_{k=0}^{\ell-1}\binom{\ell+k-1}{k}\left(\frac{x+1}{2}\right)^{\ell}\left(\frac{1-x}{2}\right)^{k},$$

the number $p_{f,\{-1,1\}(\ell,\ell)}(0)$ represents the probability of player A winning. Since this game is fair regardless of ℓ for players A and B, it holds that

$$p_{f,\{-1,1\}(\ell,\ell)}(0) = \frac{1}{2}, \ \ell \in \mathbf{N}.$$

(2) Analogously we can give an intuitive probabilistic explanation of Theorem 1.3.9. Without loss of generality, we can assume that

$$f(x) = \begin{cases} 1 & x \in \left[\frac{n-m}{n+m}, \infty\right) \\ 0 & x \in \left(-\infty, \frac{n-m}{n+m}\right) \end{cases}.$$

Two players A, B play the following game:

A coin is tossed repeatedly. The probability of heads on any toss is $p = \frac{n}{n+m}$ and the probability of tails on any toss is $1 - p = \frac{m}{n+m}$. If the coin lands heads up, then player A goes forward 1 spaces. If the coin lands on the reverse, then player B goes forward 1 spaces. The distance from the start to player A's goal is $n\ell$ spaces. The distance from the start to player B's goal is $m\ell$ spaces. The player who reaches the first, wins.

Since $p_{f,\{-1,1\}(n\ell,m\ell)}$ is expressed as

$$p_{f,\{-1,1\}(n\ell,m\ell)}(x) = \sum_{k=0}^{m\ell-1} \binom{n\ell+k-1}{k} \left(\frac{x+1}{2}\right)^{n\ell} \left(\frac{1-x}{2}\right)^{k},$$

the number $p_{f,\{-1,1\}(n\ell,m\ell)}\left(\frac{n-m}{n+m}\right)$ represents the probability of player A winning. When we toss the coin $(n+m)\ell$ times, we can expect that player A is near A's goal and player B is near B's goal. Therefore, it holds that

$$\lim_{\ell\to\infty} p_{f,\{-1,1\}(n\ell,m\ell)}\left(\frac{n-m}{n+m}\right) = \frac{1}{2}.$$

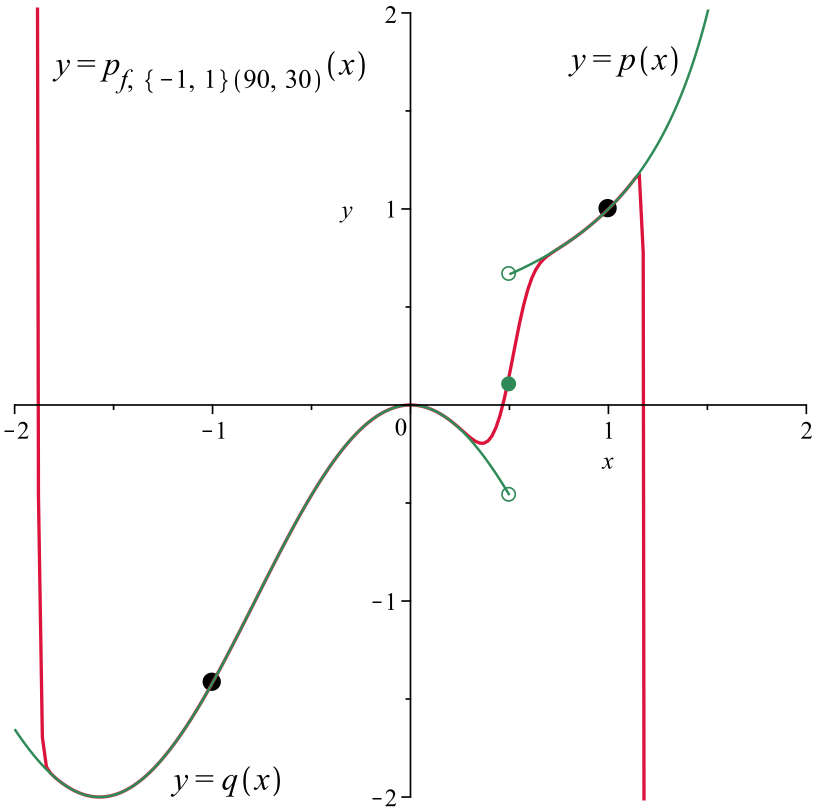
Corollary 3.3.2. *Let m, n be positive integers. Let δ_1 be a real number with $\delta_1 > \frac{n-m}{n+m} - (-1)$ and δ_2 a real number with $\delta_2 > 1 - \frac{n-m}{n+m}$, where $\frac{n-m}{n+m}$ is the point which divides the interval $[-1, 1]$ in the ratio $n : m$. Let f be a piecewise analytic function*

$$f(x) = \begin{cases} p(x) & x \in \left(\frac{n-m}{n+m}, \infty\right) \\ \frac{p\left(\frac{n-m}{n+m}\right)+q\left(\frac{n-m}{n+m}\right)}{2} & x = \frac{n-m}{n+m} \\ q(x) & x \in \left(-\infty, \frac{n-m}{n+m}\right) \end{cases}$$

such that f is equal to an analytic function p on $\left[\frac{n-m}{n+m}, \infty\right)$ which has the Taylor expansion of p about 1 on $(1-\delta_2, 1+\delta_2)$, and f is equal to an analytic function q on $\left(-\infty, \frac{n-m}{n+m}\right)$ which has the Taylor expansion of q about -1 on $(-1-\delta_1, -1+\delta_1)$. Let $p_{f,\{-1,1\}(n\ell,m\ell)}, \ell \in \mathbf{N}$ be the Hermite interpolating polynomials to f at $-1, 1$ with multiplicities $n\ell, m\ell$. Let α be the real number with $\alpha < -1$ and $|(\alpha+1)^n(\alpha-1)^m| = \frac{2^{n+m}\cdot n^n\cdot m^m}{(n+m)^{n+m}}$ and β the real number *with* $\beta > 1$ *and* $|(\beta+1)^n(\beta-1)^m| = \frac{2^{n+m} \cdot n^n \cdot m^m}{(n+m)^{n+m}}$. *Then, for each* $x \in [\alpha, \beta]$, *there exists a positive number* C *such that*

$$\left|p_{f,\{-1,1\}(n\ell,m\ell)}(x) - f(x)\right| \leq \frac{C}{\sqrt{\ell}} \text{ for all } \ell \in \mathbf{N},$$

that is, f *has the two point Taylor expansion about* $-1, 1$ *with multiplicity weight* (n, m) *on* $[\alpha, \beta]$.



$$p(x) = \frac{1}{1-(x-1)},\ q(x) = \cos 2x - 1,\ m = 1,\ n = 3,\ \ell = 30$$

Chapter 4

Termwise differentiation of two point Taylor expansion

4.1 Main Result

The purpose of this chapter is to prove the following theorem.

Theorem 4.1.1. *Let m, n be positive integers. Let f be a piecewise polynomial function*

$$f(x) = \begin{cases} p(x) & x \in \left[\frac{n-m}{n+m}, \infty\right) \\ q(x) & x \in \left(-\infty, \frac{n-m}{n+m}\right) \end{cases}$$

such that p and q are polynomials of degree at most N. Let $p_{f,\{-1,1\}(n\ell,m\ell)}, \ell \in \mathbf{N}$ be the Hermite interpolating polynomials to f at $-1, 1$ with multiplicities $n\ell, m\ell$. Let α be the real number with $\alpha < -1$ and $|(\alpha+1)^n(\alpha-1)^m| = \frac{2^{n+m}\cdot n^n\cdot m^m}{(n+m)^{n+m}}$ and β the real number with $\beta > 1$ and $|(\beta+1)^n(\beta-1)^m| = \frac{2^{n+m}\cdot n^n\cdot m^m}{(n+m)^{n+m}}$. Then, it holds that, for any given positive integer k

$$\lim_{\ell\to\infty} p^{(k)}_{f,\{-1,1\}(n\ell,m\ell)}(x) = f^{(k)}(x) \text{ for all } x \in \left(\alpha, \frac{n-m}{n+m}\right) \cup \left(\frac{n-m}{n+m}, \beta\right).$$

4.2 Divided differences of a truncated power function

Before proving Theorem 4.1.1, we show three propositions about divided differences of a truncated power function.

Proposition 4.2.1. *Let a be a real number. Let m, n be positive integers and N an integer with $N \geq 0$. Then, it holds that*

$$\frac{1}{(m-1)!}\left(\frac{1}{(x+a)^{n-N}}\right)^{(m-1)}\Bigg|_{x=a} = \frac{(-1)^{m-1}}{(m-1)!}(n-N)\cdots\left\{(n-N)+(m-2)\right\}\frac{1}{(2a)^{n-N+m-1}}.$$

Proof. Since we have

$$
\begin{aligned}
&\left(\frac{1}{(x+a)^{n-N}}\right)^{(m-1)} \\
&= \left\{(x+a)^{-(n-N)}\right\}^{(m-1)} \\
&= \left\{-(n-N)\right\}\left\{-(n-N)-1\right\}\cdots\left\{-(n-N)-(m-1)+1\right\}(x+a)^{-(n-N)-(m-1)} \\
&= (-1)^{m-1}(n-N)\cdots\left\{(n-N)+(m-1)-1\right\}\frac{1}{(x+a)^{(n-N)+m-1}},
\end{aligned}
$$

we obtain

$$
\frac{1}{(m-1)!}\left(\frac{1}{(x+a)^{n-N}}\right)^{(m-1)}\Bigg|_{x=a} = \frac{(-1)^{m-1}}{(m-1)!}(n-N)\cdots\left\{(n-N)+(m-2)\right\}\frac{1}{(2a)^{n-N+m-1}}.
$$

□

Proposition 4.2.2. *Let k, m, n be positive integers and N an integer with $N \geq 0$. Let $f_N(x)$ and $g_N(x)$ be functions given by*

$$
f_N(x) = \begin{cases} (x+1)^N & , \ x \geq \dfrac{n-m}{n+m} \\[2ex] 0 & , \ x < \dfrac{n-m}{n+m} \end{cases}
$$

and

$$
g_N(x) = \begin{cases} 0 & , \ x \geq \dfrac{n-m}{n+m} \\[2ex] (x-1)^N & , \ x < \dfrac{n-m}{n+m} \end{cases} .
$$

Then, the following hold:

(1) *For each $j = 1, \ldots, n$,*

$$
\begin{aligned}
&f_N[-1, 1; nk+j, mk] \\
&= \frac{(-1)^{mk-1}}{(mk-1)!}(nk+j-N)\cdots\left\{(nk+j-N)+(mk-2)\right\}\frac{1}{2^{(n+m)k+j-N-1}}.
\end{aligned}
$$

(2) *For each $j = 1, \ldots, m$,*

$$
\begin{aligned}
&f_N[-1, 1; n(k+1), mk+j] \\
&= \frac{(-1)^{mk+j-1}}{(mk+j-1)!}(nk+n-N)\cdots\left\{(nk+n-N)+(mk+j-2)\right\}\frac{1}{2^{n(k+1)-N+mk+j-1}}.
\end{aligned}
$$

(3) *For each $j = 1, \ldots, n$,*

$$
\begin{aligned}
&g_N[-1, 1; nk+j, mk] \\
&= \frac{(-1)^{mk-N}}{(nk+j-1)!}(mk-N)\cdots\left\{(mk-N)+(nk+j-2)\right\}\frac{1}{2^{(n+m)k+j-N-1}}.
\end{aligned}
$$

(4) *For each* $j = 1, \ldots, m$,

$$g_N[-1, 1; n(k+1), mk+j]$$
$$= \frac{(-1)^{mk+j-N}}{(nk+n-1)!}(mk+j-N)\cdots\{(mk+j-N)+(nk+n-2)\}\frac{1}{2^{mk+j-N+nk+n-1}}.$$

Proof. We prove only (1), (2) since we can prove (3), (4) similarly to (1), (2), respectively.
First, we prove (1). From Proposition 3.2.1, we get

$$\begin{aligned} f_N[-1, 1; n, m] &= \frac{1}{(n-1)!}\left(\frac{f_N(x)}{(x-1)^m}\right)^{(n-1)}\Bigg|_{x=-1} + \frac{1}{(m-1)!}\left(\frac{f_N(x)}{(x+1)^n}\right)^{(m-1)}\Bigg|_{x=1} \\ &= \frac{1}{(m-1)!}\left(\frac{1}{(x+1)^{n-N}}\right)^{(m-1)}\Bigg|_{x=1}. \end{aligned}$$

Therefore, from Proposition 4.2.1, we have

$$\begin{aligned} & f_N[-1, 1; nk+j, mk] \\ &= \frac{(-1)^{mk-1}}{(mk-1)!}(nk+j-N)\cdots\{(nk+j-N)+(mk-2)\}\frac{1}{2^{nk+j-N+mk-1}} \\ &= \frac{(-1)^{mk-1}}{(mk-1)!}(nk+j-N)\cdots\{(nk+j-N)+(mk-2)\}\frac{1}{2^{(n+m)k+j-N-1}}. \end{aligned}$$

Next, we prove (2). Similarly to (1), we have

$$\begin{aligned} & f_N[-1, 1; n(k+1), mk+j] \\ &= \frac{(-1)^{mk+j-1}}{(mk+j-1)!}\frac{(n(k+1)-N)\cdots\{(n(k+1)-N)+(mk+j-2)\}}{2^{n(k+1)-N+mk+j-1}} \\ &= \frac{(-1)^{mk+j-1}}{(mk+j-1)!}\frac{(nk+n-N)\cdots\{(nk+n-N)+(mk+j-2)\}}{2^{n(k+1)-N+mk+j-1}}. \end{aligned}$$

□

Proposition 4.2.3. *Let* k, m, n *be positive integers and* N *an integer with* $N \geq 0$. *Let* $f_N(x)$ *and* $g_N(x)$ *be functions given by*

$$f_N(x) = \begin{cases} (x+1)^N & , \ x \geq \dfrac{n-m}{n+m} \\ 0 & , \ x < \dfrac{n-m}{n+m} \end{cases}$$

and

$$g_N(x) = \begin{cases} 0 & , \ x \geq \dfrac{n-m}{n+m} \\ (x-1)^N & , \ x < \dfrac{n-m}{n+m} \end{cases}.$$

Then, the following hold:

(1) *For each* $j = 1, \ldots, n$,

$$\lim_{k\to\infty}\left|\frac{f_N[-1,1;nk+j,mk]}{f_N[-1,1;n(k+1)+j,m(k+1)]}\right| = \frac{m^m\cdot n^n\cdot 2^{n+m}}{(n+m)^{n+m}}.$$

(2) *For each* $j = 1, \ldots, m$,

$$\lim_{k\to\infty}\left|\frac{f_N[-1,1;n(k+1),mk+j]}{f_N[-1,1;n(k+2),m(k+1)+j]}\right| = \frac{m^m\cdot n^n\cdot 2^{n+m}}{(n+m)^{n+m}}.$$

(3) *For each* $j = 1, \ldots, n$,

$$\lim_{k\to\infty}\left|\frac{g_N[-1,1;nk+j,mk]}{g_N[-1,1;n(k+1)+j,m(k+1)]}\right| = \frac{m^m\cdot n^n\cdot 2^{n+m}}{(n+m)^{n+m}}.$$

(4) *For each* $j = 1, \ldots, m$,

$$\lim_{k\to\infty}\left|\frac{g_N[-1,1;n(k+1),mk+j]}{g_N[-1,1;n(k+2),m(k+1)+j]}\right| = \frac{m^m\cdot n^n\cdot 2^{n+m}}{(n+m)^{n+m}}.$$

Proof. We prove only (1), (2) since we can prove (3), (4) similarly to (1), (2), respectively.

First, we prove (1). From Proposition 4.2.2, we get for each $j = 1, \ldots, n$, and for sufficiently large k,

$$\begin{aligned}
&\left|\frac{f_N[-1,1;nk+j,mk]}{f_N[-1,1;n(k+1)+j,m(k+1)]}\right| \\
&= \frac{\dfrac{(nk+j-N)\cdots\big\{(nk+j-N)+(mk-2)\big\}}{(mk-1)!}\dfrac{1}{2^{(n+m)k+j-N-1}}}{\dfrac{(n(k+1)+j-N)\cdots\big\{(n(k+1)+j-N)+(m(k+1)-2)\big\}}{(m(k+1)-1)!}\dfrac{1}{2^{(n+m)(k+1)+j-N-1}}} \\
&= \frac{\overbrace{(mk-1+m)\cdots(mk)}^{m}\cdot\overbrace{(nk+j-N)\cdots(nk+j-N+(n-1))}^{n}\cdot 2^{n+m}}{\underbrace{(nk+j-N+(mk-2)+1)\cdots(nk+j-N+(mk-2)+(n+m))}_{n+m}} \\
&\to \frac{m^m\cdot n^n\cdot 2^{n+m}}{(n+m)^{n+m}}\ (k\to\infty).
\end{aligned}$$

Next, we prove (2). From Proposition 4.2.2, we obtain for each $j = 1, \ldots, m$, and for sufficiently large k,

$$\left|\frac{f_N[-1,1;n(k+1),mk+j]}{f_N[-1,1;n(k+2),m(k+1)+j]}\right|$$

$$= \frac{\frac{(nk+n-N)\cdots\left\{(nk+n-N)+(mk+j-2)\right\}}{(mk+j-1)!}\frac{1}{2^{n(k+1)-N+mk+j-1}}}{\frac{(n(k+1)+n-N)\cdots\left\{(n(k+1)+n-N)+(m(k+1)+j-2)\right\}}{(m(k+1)+j-1)!}\frac{1}{2^{n(k+2)-N+m(k+1)+j-1}}}$$

$$= \frac{\overbrace{(mk+j-1+m)\cdots(mk+j)}^{m}\cdot\overbrace{(nk+n-N)\cdots(nk+n-N+n-1)}^{n}\cdot 2^{n+m}}{\underbrace{(nk+n-N+(mk+j-2)+1)\cdots(nk+n-N+(mk+j-2)+n+m)}_{n+m}}$$

$$\to \frac{m^m\cdot n^n\cdot 2^{n+m}}{(n+m)^{n+m}}\ (k\to\infty).$$

□

4.3 Proof of Theorem 4.1.1

We need to prepare four propositions to show Theorem 4.1.1.

Proposition 4.3.1. *Let m,n be positive integers. Let $\{f_N(x)\}_{N\geq 0}$ be the sequence of functions defined by*

$$f_N(x) = \begin{cases} (x+1)^N & , \ x \geq \dfrac{n-m}{n+m} \\ 0 & , \ x < \dfrac{n-m}{n+m} \end{cases}$$

and $\{g_N(x)\}_{N\geq 0}$ the sequence of functions defined by

$$g_N(x) = \begin{cases} 0 & , \ x \geq \dfrac{n-m}{n+m} \\ (x-1)^N & , \ x < \dfrac{n-m}{n+m} \end{cases}.$$

Let α be the real number with $\alpha < -1$ and $|(\alpha+1)^n(\alpha-1)^m| = \frac{2^{n+m}\cdot n^n\cdot m^m}{(n+m)^{n+m}}$ and β the real number with $\beta > 1$ and $|(\beta+1)^n(\beta-1)^m| = \frac{2^{n+m}\cdot n^n\cdot m^m}{(n+m)^{n+m}}$. Then, the following hold:
(1) *For each $N = 0,1,2,\ldots$, the series*

$$F_N(x) = \sum_{j=1}^{n}(x+1)^{j-1}\sum_{k=0}^{\infty} f_N[-1,1;nk+j,mk]\left\{(x+1)^n(x-1)^m\right\}^k$$

$$+\sum_{j=1}^{m}(x+1)^n(x-1)^{j-1}\sum_{k=0}^{\infty} f_N[-1,1;n(k+1),mk+j]\left\{(x+1)^n(x-1)^m\right\}^k$$

converges for $x \in \left(\alpha, \frac{n-m}{n+m}\right) \cup \left(\frac{n-m}{n+m}, \beta\right)$.

(2) *For each* $N = 0, 1, 2, \ldots$, *the series*

$$G_N(x) = \sum_{j=1}^{n} (x+1)^{j-1} \sum_{k=0}^{\infty} g_N[-1, 1; nk+j, mk] \left\{(x+1)^n (x-1)^m\right\}^k$$
$$+ \sum_{j=1}^{m} (x+1)^n (x-1)^{j-1} \sum_{k=0}^{\infty} g_N[-1, 1; n(k+1), mk+j] \left\{(x+1)^n (x-1)^m\right\}^k$$

converges for $x \in \left(\alpha, \frac{n-m}{n+m}\right) \cup \left(\frac{n-m}{n+m}, \beta\right)$.

Proof. We prove only (1) since we can prove (2) similarly to (1). From Proposition 4.2.3, we have for each $N = 0, 1, \ldots$ and for each $j = 1, \ldots, n$, the power series

$$\sum_{k=0}^{\infty} f_N[-1, 1; nk+j, mk] t^k$$

converges on the interval $\left(-\frac{n^n \cdot m^m \cdot 2^{n+m}}{(n+m)^{n+m}}, \frac{n^n \cdot m^m \cdot 2^{n+m}}{(n+m)^{n+m}}\right)$ and we have for each $N = 0, 1, \ldots$ and for each $j = 1, \ldots, m$, the power series

$$\sum_{k=0}^{\infty} f_N[-1, 1; n(k+1), mk+j] t^k$$

converges on the interval $\left(-\frac{n^n \cdot m^m \cdot 2^{n+m}}{(n+m)^{n+m}}, \frac{n^n \cdot m^m \cdot 2^{n+m}}{(n+m)^{n+m}}\right)$. Therefore, we see that for each $N = 0, 1, \ldots$ and for each $j = 1, \ldots, n$, the series

$$(x+1)^{j-1} \sum_{k=0}^{\infty} f_N[-1, 1; nk+j, mk] \left\{(x+1)^n (x-1)^m\right\}^k$$

converges for $x \in \left(\alpha, \frac{n-m}{n+m}\right) \cup \left(\frac{n-m}{n+m}, \beta\right)$ and we observe that for each $N = 0, 1, \ldots$ and for each $j = 1, \ldots, m$, the series

$$(x+1)^n (x-1)^{j-1} \sum_{k=0}^{\infty} f_N[-1, 1; n(k+1), mk+j] \left\{(x+1)^n (x-1)^m\right\}^k$$

converges for $x \in \left(\alpha, \frac{n-m}{n+m}\right) \cup \left(\frac{n-m}{n+m}, \beta\right)$. Hence, we see that $F_N(x)$, $N = 0, 1, 2, \ldots$, converge for $x \in \left(\alpha, \frac{n-m}{n+m}\right) \cup \left(\frac{n-m}{n+m}, \beta\right)$.

$\square$

Proposition 4.3.2. *Let* ℓ, m, n *be positive integers and* L_1, L_2 *integers with* $L_1 \geq 0$ *and* $L_2 \geq 0$. *Let* $\{f_N(x)\}_{N \geq 0}$ *be the sequence of functions defined by*

$$f_N(x) = \begin{cases} (x+1)^N & , \ x \geq \dfrac{n-m}{n+m} \\ 0 & , \ x < \dfrac{n-m}{n+m} \end{cases}$$

and $\{g_N(x)\}_{N\geq 0}$ *the sequence of functions defined by*

$$g_N(x) = \begin{cases} 0 & , \ x \geq \dfrac{n-m}{n+m} \\ (x-1)^N & , \ x < \dfrac{n-m}{n+m} \end{cases} .$$

Let $f(x)$ *be a function given by*

$$f(x) = \begin{cases} \displaystyle\sum_{i=0}^{L_1} a_i(x+1)^i & , \ x \geq \dfrac{n-m}{n+m} \\ \displaystyle\sum_{i=0}^{L_2} b_i(x-1)^i & , \ x < \dfrac{n-m}{n+m} \end{cases} ,$$

where $a_0, \ldots, a_{L_1}, b_0, \ldots, b_{L_2}$ *are real numbers. Let* $p_{f,\{-1,1\}(n\ell,m\ell)}(x), \ell \in \mathbf{N}$ *be the Hermite interpolating polynomials to* f *at* $-1, 1$ *with multiplicities* $n\ell, m\ell$. *Let* $\{F_N(x)\}_{N=0}^{L_1}$ *be the sequence of functions on the set* $\left(\alpha, \frac{n-m}{n+m}\right) \cup \left(\frac{n-m}{n+m}, \beta\right)$ *defined by*

$$\begin{aligned} F_N(x) &= \sum_{j=1}^{n}(x+1)^{j-1}\sum_{k=0}^{\infty} f_N[-1,1;nk+j,mk]\left\{(x+1)^n(x-1)^m\right\}^k \\ &\quad + \sum_{j=1}^{m}(x+1)^n(x-1)^{j-1}\sum_{k=0}^{\infty} f_N[-1,1;n(k+1),mk+j]\left\{(x+1)^n(x-1)^m\right\}^k \end{aligned}$$

and $\{G_N(x)\}_{N=0}^{L_2}$ *the sequence of functions on the set* $\left(\alpha, \frac{n-m}{n+m}\right) \cup \left(\frac{n-m}{n+m}, \beta\right)$ *defined by*

$$\begin{aligned} G_N(x) &= \sum_{j=1}^{n}(x+1)^{j-1}\sum_{k=0}^{\infty} g_N[-1,1;nk+j,mk]\left\{(x+1)^n(x-1)^m\right\}^k \\ &\quad + \sum_{j=1}^{m}(x+1)^n(x-1)^{j-1}\sum_{k=0}^{\infty} g_N[-1,1;n(k+1),mk+j]\left\{(x+1)^n(x-1)^m\right\}^k . \end{aligned}$$

Let $S(x)$ *be the function on the set* $\left(\alpha, \frac{n-m}{n+m}\right) \cup \left(\frac{n-m}{n+m}, \beta\right)$ *defined by*

$$S(x) = \sum_{i=0}^{L_1} a_i F_i(x) + \sum_{i=0}^{L_2} b_i G_i(x).$$

Let α *be the real number with* $\alpha < -1$ *and* $|(\alpha+1)^n(\alpha-1)^m| = \frac{2^{n+m}\cdot n^n \cdot m^m}{(n+m)^{n+m}}$ *and* β *the real number with* $\beta > 1$ *and* $|(\beta+1)^n(\beta-1)^m| = \frac{2^{n+m}\cdot n^n \cdot m^m}{(n+m)^{n+m}}$. *Then, for each* $k = 0, 1, 2, \ldots$, *it holds that*

$$\lim_{\ell\to\infty} p^{(k)}_{f,\{-1,1\}(n\ell,m\ell)}(x) = S^{(k)}(x) \text{ for all } x \in \left(\alpha, \frac{n-m}{n+m}\right) \cup \left(\frac{n-m}{n+m}, \beta\right).$$

Proof. For each $N = 0, 1, 2, \ldots$, let $p_{N,\ell}(x), \ell \in \mathbf{N}$ be the Hermite interpolating polynomials to f_N at $-1, 1$ with multiplicities $n\ell, m\ell$ and $q_{N,\ell}(x), \ell \in \mathbf{N}$ the Hermite interpolating polynomials to g_N at $-1, 1$ with multiplicities $n\ell, m\ell$. Then, from Proposition 2.2.2, we obtain the following expressions of $p_{N,\ell}(x)$ and $q_{N,\ell}(x)$:

$$\begin{aligned} & p_{N,\ell}(x) \\ &= \sum_{j=1}^{n} (x+1)^{j-1} \sum_{k=0}^{\ell-1} f_N[-1, 1; nk+j, mk] \left\{(x+1)^n (x-1)^m\right\}^k \\ &\quad + \sum_{j=1}^{m} (x+1)^n (x-1)^{j-1} \sum_{k=0}^{\ell-1} f_N[-1, 1; n(k+1), mk+j] \left\{(x+1)^n (x-1)^m\right\}^k, \\ & q_{N,\ell}(x) \\ &= \sum_{j=1}^{n} (x+1)^{j-1} \sum_{k=0}^{\ell-1} g_N[-1, 1; nk+j, mk] \left\{(x+1)^n (x-1)^m\right\}^k \\ &\quad + \sum_{j=1}^{m} (x+1)^n (x-1)^{j-1} \sum_{k=0}^{\ell-1} g_N[-1, 1; n(k+1), mk+j] \left\{(x+1)^n (x-1)^m\right\}^k. \end{aligned}$$

Also, from Proposition 4.3.1, we see that $F_N(x)$, $N = 0, 1, \ldots$ are infinitely differentiable on the set $\left(\alpha, \frac{n-m}{n+m}\right) \cup \left(\frac{n-m}{n+m}, \beta\right)$. Therefore, from the definition of $F_N(x)$, we have for each $i = 0, 1, \ldots$ and for each $k = 0, 1, \ldots$,

$$\lim_{\ell \to \infty} p_{i,\ell}^{(k)}(x) = F_i^{(k)}(x), \; x \in \left(\alpha, \frac{n-m}{n+m}\right) \cup \left(\frac{n-m}{n+m}, \beta\right).$$

Similarly, we obtain for each $i = 0, 1, \ldots$ and for each $k = 0, 1, \ldots$,

$$\lim_{\ell \to \infty} q_{i,\ell}^{(k)}(x) = G_i^{(k)}(x), \; x \in \left(\alpha, \frac{n-m}{n+m}\right) \cup \left(\frac{n-m}{n+m}, \beta\right).$$

Hence, we have for each $x \in \left(\alpha, \frac{n-m}{n+m}\right) \cup \left(\frac{n-m}{n+m}, \beta\right)$,

$$\begin{aligned} \lim_{\ell \to \infty} p_{f,\{-1,1\}(n\ell, m\ell)}^{(k)}(x) &= \lim_{\ell \to \infty} \left(\sum_{i=0}^{L_1} a_i p_{i,\ell}^{(k)}(x) + \sum_{i=0}^{L_2} b_i q_{i,\ell}^{(k)}(x) \right) \\ &= \sum_{i=0}^{L_1} a_i F_i^{(k)}(x) + \sum_{i=0}^{L_2} b_i G_i^{(k)}(x) \\ &= S^{(k)}(x). \end{aligned}$$

□

Proposition 4.3.3. *Let $f(x), S(x)$ be the functions defined in Proposition 4.3.2. Then, it holds that, for each $j = 0, 1, \ldots$,*

$$S^{(j)}(-1) = f^{(j)}(-1)$$

and

$$S^{(j)}(1) = f^{(j)}(1).$$

Proof. We fix $\ell \in \mathbf{N}$. Let $f_N(x), g_N(x), p_{f,\{-1,1\}(n\ell,m\ell)}(x)$ be functions defined in Proposition 4.3.2. For each $N = 0, 1, 2, \ldots$, let $p_{N,\ell}(x)$ be the Hermite interpolating polynomial to f_N at $-1, 1$ with multiplicities $n\ell, m\ell$ and $q_{N,\ell}(x)$ the Hermite interpolating polynomial to g_N at $-1, 1$ with multiplicities $n\ell, m\ell$. Also, let $p_{*,N,\ell}(x), q_{*,N,\ell}(x)$ be functions on the set $\left(\alpha, \frac{n-m}{n+m}\right) \cup \left(\frac{n-m}{n+m}, \beta\right)$ defined by

$$\begin{aligned}
&p_{*,N,\ell}(x)\\
&= F_N(x) - p_{N,\ell}(x)\\
&= \sum_{j=1}^{n}(x+1)^{j-1}\sum_{k=\ell}^{\infty} f_N[-1,1;nk+j,mk]\left\{(x+1)^n(x-1)^m\right\}^k\\
&\quad + \sum_{j=1}^{m}(x+1)^n(x-1)^{j-1}\sum_{k=\ell}^{\infty} f_N[-1,1;n(k+1),mk+j]\left\{(x+1)^n(x-1)^m\right\}^k
\end{aligned}$$

and

$$\begin{aligned}
&q_{*,N,\ell}(x)\\
&= G_N(x) - q_{N,\ell}(x)\\
&= \sum_{j=1}^{n}(x+1)^{j-1}\sum_{k=\ell}^{\infty} g_N[-1,1;nk+j,mk]\left\{(x+1)^n(x-1)^m\right\}^k\\
&\quad + \sum_{j=1}^{m}(x+1)^n(x-1)^{j-1}\sum_{k=\ell}^{\infty} g_N[-1,1;n(k+1),mk+j]\left\{(x+1)^n(x-1)^m\right\}^k.
\end{aligned}$$

Now we have

$$\begin{aligned}
S(x) &= \sum_{i=0}^{L_1} a_i F_i(x) + \sum_{i=0}^{L_2} b_i G_i(x)\\
&= \sum_{i=0}^{L_1} a_i\left(p_{i,\ell}(x) + p_{*,i,\ell}(x)\right) + \sum_{i=0}^{L_2} b_i\left(q_{i,\ell}(x) + q_{*,i,\ell}(x)\right)\\
&= \left(\sum_{i=0}^{L_1} a_i p_{i,\ell}(x) + \sum_{i=0}^{L_2} b_i q_{i,\ell}(x)\right) + \sum_{i=0}^{L_1} a_i p_{*,i,\ell}(x) + \sum_{i=0}^{L_2} b_i q_{*,i,\ell}(x)\\
&= p_{f,\{-1,1\}(n\ell,m\ell)}(x) + \sum_{i=0}^{L_1} a_i p_{*,i,\ell}(x) + \sum_{i=0}^{L_2} b_i q_{*,i,\ell}(x).
\end{aligned}$$

Since $p_{*,N,\ell}(x), q_{*,N,\ell}(x)$ are expressed as

$$
\begin{aligned}
&p_{*,N,\ell}(x)\\
&=\sum_{j=1}^{n}(x+1)^{n\ell+j-1}(x-1)^{m\ell}\sum_{k=0}^{\infty}f_N[-1,1;n(k+\ell)+j,m(k+\ell)]\left\{(x+1)^n(x-1)^m\right\}^k\\
&\quad+\sum_{j=1}^{m}(x+1)^{n\ell+n}(x-1)^{m\ell+j-1}\sum_{k=0}^{\infty}f_N[-1,1;n(k+\ell+1),m(k+\ell)+j]\\
&\quad\times\left\{(x+1)^n(x-1)^m\right\}^k
\end{aligned}
$$

and

$$
\begin{aligned}
&q_{*,N,\ell}(x)\\
&=\sum_{j=1}^{n}(x+1)^{n\ell+j-1}(x-1)^{m\ell}\sum_{k=0}^{\infty}g_N[-1,1;n(k+\ell)+j,m(k+\ell)]\left\{(x+1)^n(x-1)^m\right\}^k\\
&\quad+\sum_{j=1}^{m}(x+1)^{n\ell+n}(x-1)^{m\ell+j-1}\sum_{k=0}^{\infty}g_N[-1,1;n(k+\ell+1),m(k+\ell)+j]\\
&\quad\times\left\{(x+1)^n(x-1)^m\right\}^k,
\end{aligned}
$$

we obtain

$$p^{(j)}_{*,N,\ell}(-1)=q^{(j)}_{*,N,\ell}(-1)=0 \text{ for each } j=0,\dots,n\ell-1,$$

and

$$p^{(j)}_{*,N,\ell}(1)=q^{(j)}_{*,N,\ell}(1)=0 \text{ for each } j=0,\dots,m\ell-1.$$

Therefore, from the definition of $p_{f,\{-1,1\}(n\ell,m\ell)}(x)$, we have for each $j=0,\dots,n\ell-1$,

$$
\begin{aligned}
S^{(j)}(-1) &= p^{(j)}_{f,\{-1,1\}(n\ell,m\ell)}(-1)+\sum_{i=0}^{L_1}a_ip^{(j)}_{*,i,\ell}(-1)+\sum_{i=0}^{L_2}b_iq^{(j)}_{*,i,\ell}(-1)\\
&= f^{(j)}(-1),
\end{aligned}
$$

and we obtain for each $j=0,\dots,m\ell-1$,

$$
\begin{aligned}
S^{(j)}(1) &= p^{(j)}_{f,\{-1,1\}(n\ell,m\ell)}(1)+\sum_{i=0}^{L_1}a_ip^{(j)}_{*,i,\ell}(1)+\sum_{i=0}^{L_2}b_iq^{(j)}_{*,i,\ell}(1)\\
&= f^{(j)}(1).
\end{aligned}
$$

Since ℓ is arbitrary, we get for each $j=0,1,\dots,$

$$S^{(j)}(-1)=f^{(j)}(-1)$$

and

$$S^{(j)}(1)=f^{(j)}(1).$$

□

Proposition 4.3.4. *Let $f(x), S(x)$ be the functions defined in Proposition 4.3.2. Then, it holds that, for each $k = 0, 1, \ldots,$*

$$S^{(k)}(x) = f^{(k)}(x) \text{ for all } x \in \left(\alpha, \frac{n-m}{n+m}\right) \cup \left(\frac{n-m}{n+m}, \beta\right).$$

Proof. We fix $k \in \{0, 1, 2, \ldots\}$. We put

$$p(x) = \sum_{i=0}^{L_1} a_i (x+1)^i$$

and

$$q(x) = \sum_{i=0}^{L_2} b_i (x-1)^i.$$

Since the function $S^{(k)}(x)$ is analytic on the interval $\left(\alpha, \frac{n-m}{n+m}\right)$, there exists an $\varepsilon > 0$ such that

$$S^{(k)}(x) = \sum_{j=0}^{\infty} \frac{S^{(k+j)}(-1)}{j!}(x+1)^j \text{ for all } x \in (-1-\varepsilon, -1+\varepsilon).$$

Also, the function $q^{(k)}(x)$ can be expressed as

$$q^{(k)}(x) = \sum_{j=0}^{\infty} \frac{q^{(k+j)}(-1)}{j!}(x+1)^j.$$

Now, from Proposition 4.3.3, we have for each $j \in \{0, 1, \ldots\}$,

$$S^{(k+j)}(-1) = f^{(k+j)}(-1) = q^{(k+j)}(-1).$$

Hence, from the identity theorem, we obtain

$$S^{(k)}(x) = q^{(k)}(x) \text{ for all } x \in \left(\alpha, \frac{n-m}{n+m}\right).$$

Similarly, we have

$$S^{(k)}(x) = p^{(k)}(x) \text{ for all } x \in \left(\frac{n-m}{n+m}, \beta\right).$$

From the above, we obtain

$$S^{(k)}(x) = f^{(k)}(x)$$

for all $x \in \left(\alpha, \frac{n-m}{n+m}\right) \cup \left(\frac{n-m}{n+m}, \beta\right)$. □

Now we are in position to prove Theorem 4.1.1.

Proof of Theorem 4.1.1. Let $S(x)$ be the function defined in Proposition 4.3.2. We fix $k \in \{0, 1, 2, \ldots\}$. From Proposition 4.3.2, we have

$$\lim_{\ell \to \infty} p^{(k)}_{f,\{-1,1\}(n\ell, m\ell)}(x) = S^{(k)}(x) \text{ for all } x \in \left(\alpha, \frac{n-m}{n+m}\right) \cup \left(\frac{n-m}{n+m}, \beta\right).$$

Furthermore, from Proposition 4.3.4, we obtain

$$S^{(k)}(x) = f^{(k)}(x) \text{ for all } x \in \left(\alpha, \frac{n-m}{n+m}\right) \cup \left(\frac{n-m}{n+m}, \beta\right).$$

Hence, we get

$$\lim_{\ell \to \infty} p^{(k)}_{f,\{-1,1\}(n\ell,m\ell)}(x) = f^{(k)}(x) \text{ for all } x \in \left(\alpha, \frac{n-m}{n+m}\right) \cup \left(\frac{n-m}{n+m}, \beta\right).$$

$\square$

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