Correction of Sraffa's Imaginary Experiment*

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In section 37 of *Production of Commodities by Means of Commodities*, Piero Sraffa demonstrates the existence of Standard commodity and Standard ratio. In 2008, Marco Lippi examined this section carefully and found a critical mistake in it. Further, Neri Salvadori scrutinized Lippi's paper and supported his view. While both these authors elucidated the existence of an error in Sraffa's proof, in this paper, we attempt to correct this problem. We are led to the same conclusion as Lippi and Salvadori's. Therefore, we suggest a modification to Sraffa's proof.

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1 INTRODUCTION

Piero Sraffa begins section 37 as follows: "That any actual economic system of the type we have been considering can always be transformed into a Standard system may be shown by an *imaginary experiment*"[1)]

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1) Sraffa [8] p.26.

(italics mine). In 2008, Marco Lippi examined the imaginary experiment and argued that an actual economy cannot always be translated into a Standard system through it and hence suggested its modification[2]. Neri Salvadori, who investigated Lippi's paper, confirmed that there existed a deficit in Sraffa's argument and proposed another modification.[3]

In this paper, we examine Sraffa's imaginary experiment, clarify the part where Sraffa makes a mistake, and modify his imaginary experiment. The rest of the paper is organized as follows. In section 2, we examine the intention of the imaginary experiment. In section 3, we transform Sraffa's literal explanation of the imaginary experiment into a mathematical one, after which we inquire this in detail. In section 4, we clarify the problem with the imaginary experiment. In section 5, we propose a correction to the experiment. In section 6, we examine the range of application for our correction.

2 THE OBJECT OF THE IMAGINARY EXPERIMENT

We explicate Sraffa's method of the proof before investigating the imaginary experiment. The subject of the imaginary experiment is an actual economic system, which Sraffa defines as follows: "(any actual economic) system is assumed to be in self-replacing state, $A_a + A_b \cdots + A_k \leqslant A; B_a + B_b \cdots + B_k \leqslant B; \cdots K_a + K_b \cdots + K_k \leqslant K$; that is to say the quantity produced of each commodity is *at least* equal to the quantity of it which is used up in all branches of production together"[4] (addition mine). This passage can be represented as below, where an actual economic system belongs to a set$\{\alpha \boldsymbol{x} \geq \boldsymbol{A}\boldsymbol{x}\}$. The notations are as follows, $\boldsymbol{A} = (a_{ij})$ $i, j = 1 \dots n$: input matrix, $\boldsymbol{x} = (x_i)$ $i = 1 \dots n$: output vector, α: real

2) Lippi [2]

3) Salvador [7]

4) Sraffa [8]pp.6-7.

value parameter, which satisfies $\alpha(\boldsymbol{x}) \geqq \max_{x_i \neq 0} (A\boldsymbol{x})_i / x_i$.

Let us consider $\{\alpha(\boldsymbol{x})\}$, the collection of all such real value α that satisfies the set$\{\alpha\boldsymbol{x} \geq A\boldsymbol{x}\}$. The set $\{\alpha(\boldsymbol{x})\}$ is bounded below[5)] and there exists a lower limit. Now, we examine the sequence $\{\alpha(\boldsymbol{x})\}$, which varies with $\boldsymbol{x}$. If this sequence is a monotonic decreasing sequence, it converges to the lower limit α^*, that is, $\alpha(\boldsymbol{x}) \to \alpha^*$. This means that any real economic system can converge to $\alpha^*\boldsymbol{x}^* = A\boldsymbol{x}^*$, where variable $\boldsymbol{x}^*$ is the corresponding vector to the α^*. This equation means the Standard system. Our problem is to find the method of making $\{\alpha(\boldsymbol{x})\}$a monotonic decreasing sequence (see figure 1), which Sraffa tried through his imaginary experiment. In what follows, we examine whether Sraffa's attempt succeeds.

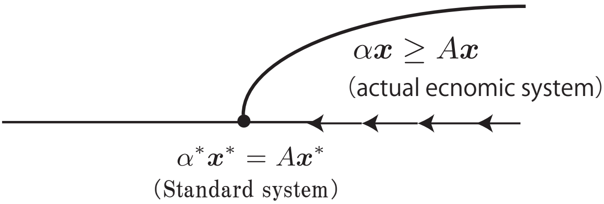


figure 1

3 IMAGINARY EXPERIMENT

Sraffa accounts for his imaginary experiment in literary terms[6)]. We translate it into mathematical terms in order to clarify the pivotal factors. We divide section 37 into four, and translate each step into mathematical terms.

5) The proof, which is due to Minc [4], is given in Mathematical Appendix A.

6) See Appendix (Primary sources).

3.1 Step1

■ Sraffa's original account[I]

> We start by adjusting the proportions of the industries of the system in such a way that of each basic commodity is a larger quantity is produced than is strictly necessary for replacement.

■ mathematical interpretation[I] Let's choose $\alpha^0 > 0$ such that

(1) $\quad A\boldsymbol{x}^0 < \alpha^0 \boldsymbol{x}^0,$

where α^0 is any real value satisfying $\alpha^0 > \max_i \dfrac{(A\boldsymbol{x}^{\mathbf{0}})_i}{x_i^0}$.

3.2 Step2

■ Sraffa's original account[II]

> Let us next imagine gradually to reduce by means of successive small proportionate cuts the product of all industries, ... the cuts reduce the production of any one commodity to the minimum level required for replacement, ...

■ mathematical interpretation[II] Let us choose $\alpha^1 > 0$ such that

(2) $\quad A\boldsymbol{x}^0 \leq \alpha^1 \boldsymbol{x}^0,$

where $\alpha^1 = \max_i \dfrac{(A\boldsymbol{x}^{\mathbf{0}})_i}{x_i^0}$.

3.3 Step3

■ Sraffa's original account[III]

> we readjust the proportions of the industries so that there should again be a surplus of each product exists for each product ..... . This is always feasible so long as there is a surplus of some commodities and a deficit of none.

■ **mathematical interpretation[III]** Let $C = \frac{A}{\| A\boldsymbol{x}^0 \|_\infty}$ [7]. We multiply the inequality (2) by C to the left. As C is a positive matrix, we get[8]

(3) $CA\boldsymbol{x}^0 < \alpha^1 C\boldsymbol{x}^0$.

Now let $\boldsymbol{x}^1 = C\boldsymbol{x}^0$, this is written as follows;

(4) $A\boldsymbol{x}^1 < \alpha^1 \boldsymbol{x}^1$.

3.4 Step4

■ **Sraffa's original account[IV]**

> We continue with such an alternation of proportionate cuts with the re-establishment of a surplus for each product until we reach the point where the products have been reduced to such an extent that all-round replacement is just possible without leaving anything as surplus product.
>
> Since to reach this position the products of all the industries have been cut in the same proportion we are now able to restore the original conditions of production by increasing the quantity produced in each industry by a uniform rate; we do not, on the other hand, disturb the proportions to which the industries have been brought. The uniform rate which restore the original condition is R and the proportions attained by the industries are the proportions of the Standard system.

■ **mathematical interpretation[IV]** By repeating the same calculation from equation (2) to equation (4), we reach q system and get the Standard system:

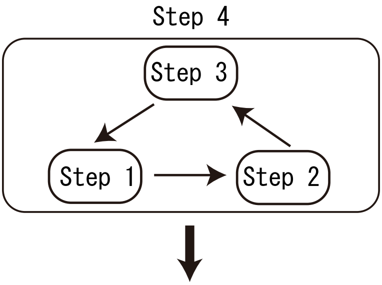
(5) $A\boldsymbol{x}^*(1 + R) = \boldsymbol{x}^*$.

7) $\| A\boldsymbol{x}^0 \|_\infty$ is $\max\{|(A\boldsymbol{x}^0)_1|, |(A\boldsymbol{x}^0)_2|, \ldots, |(A\boldsymbol{x}^0)_n|\}$.

8) Let P be a positive matrix and y be nonnegative vector, Py is positive.

3.5 Summary

The imaginary experiment is summarized in figure 2. See figure 2.



$$A\boldsymbol{x}^*(1+R)=\boldsymbol{x}^*$$

figure 2

4 EXAMINATION OF THE IMAGINARY EXPERIMENT

The imaginary experiment is a kind of an algorithm. It circulates randomly between the steps. The problem here is to determine whether the sequence $\{\alpha^i\}$ under this algorithm is a monotonic decreasing one. Concerning α^1 and α^2, $\boldsymbol{x}^{\mathbf{0}}$ must be changed to $\boldsymbol{x}^{\mathbf{1}}$ so as to satisfy the following inequality

$$(6)\qquad \alpha^1=\max_i\frac{(A\boldsymbol{x}^{\mathbf{0}})_i}{x_i^0}>\max_i\frac{(A\boldsymbol{x}^{\mathbf{1}})_i}{x_i^1}=\alpha^2.$$

Sraffa does not mention this procedure. While he seems to think that any sequence can satisfy this condition, it is incorrect. With respect to that point Salvadori points out as follows [9]: "the algorithm is not well defined since there are infinite ways to define $\boldsymbol{x}_t$. Completing the definition of the algorithm means defining a function $\phi(\boldsymbol{q})$ such that $\boldsymbol{x}_t=\phi(\boldsymbol{x}_{t-1})$, each t". Lippi and Salvadori argue that while equation (2) is clearly defined, concerning equation (4), the rule to indicate this equation is not mentioned. Consequently, one can not exclude the case in which a real economy can not translate into the Standard system.

9) Salvadori [7] p.254.

5 AMENDMENT

5.1 Proposal

We suppose that A is a positive matrix. Equation(2) multiplied by $A > 0$ from left is

$$A^2\boldsymbol{x}^{\mathbf{0}} < \alpha^1 A\boldsymbol{x}^{\mathbf{0}}.$$

Further, we let $\boldsymbol{x}^{\mathbf{1}} = A\boldsymbol{x}^{\mathbf{0}}$; then, the $\boldsymbol{x}^{\mathbf{1}}$ satisfying equation (4) is determined.

Next to translate equation(4) into equation(2), we substitute $\alpha^2 = \max_i \frac{(A\boldsymbol{x}^{\mathbf{1}})_i}{x_i^1}$; then, equation(2) is given as follows: $A\boldsymbol{x}^1 \leq \alpha^2\boldsymbol{x}^1$.

Now, we examine $\alpha^1 > \alpha^2$. From its definition, $\alpha^2 = \max_i \frac{(A\boldsymbol{x}^{\mathbf{1}})_i}{x_i^1}$ $(i = 1 \ldots n)$, then $\alpha^1 > \alpha^2$.

For $\alpha^3, \alpha^4, \alpha^5, \ldots,$ similarly, multiply the positive matrix $A > 0$ in sequence, let $\boldsymbol{x}^{\boldsymbol{n}} = A\boldsymbol{x}^{\boldsymbol{n-1}}$ $(i = 2, 3, \cdots)$, and imaginary vary the quantity of production. Then we get a monotonic decreasing sequence $\{\alpha^i\}$. We call this proposal the "corrected imaginary experiment".

5.2 Numerical example

We confirm the corrected imaginary experiment considering the numerical example. We let matrix $\boldsymbol{A} = \begin{pmatrix} 0.2 & 0.3 \\ 0.5 & 0.4 \end{pmatrix}$ and we let the convergent value of $\{\alpha^i\}$ as $r(A)$. $r(A)$ is a positive eigenvalue(= Perron root) and can be calculated as $r(A) = 0.7$.

Now we let $\boldsymbol{x}^{\mathbf{0}} = \begin{pmatrix} 5 & 7 \end{pmatrix}'$ On examining the corrected imaginary experiment, we get $\alpha^1(\boldsymbol{x}^1) = 0.757\cdots$, $\alpha^2(\boldsymbol{x}^2) = 0.712\cdots$, $\alpha^3(\boldsymbol{x}^3) = 0.701\cdots$, $\alpha^4(\boldsymbol{x}^4) = 0.7002\cdots$. Then, we get the monotonic decreasing sequence

$$r(A) < \cdots \alpha^4(\boldsymbol{x}^4) < \alpha^3(\boldsymbol{x}^3) < \alpha^2(\boldsymbol{x}^2) < \alpha^1(\boldsymbol{x}^1),$$

which is converging $r(A)$. For proof of the general case, see Mathematical Appendix B[10)].

10) The proof is due to Saito [11] and Tanaka [12].

5.3 Previous studies

According to Salvadori[11], there exist several remedies to Sraffa' s imaginary experiment as follows:

Example 1 (Kurz and Salvadori)

$$\phi(\boldsymbol{q}) = \Big[\frac{\beta}{\boldsymbol{q}^T[\boldsymbol{I}-\boldsymbol{A}]^{-1}\boldsymbol{l}}\boldsymbol{q}^T[\boldsymbol{I}-\boldsymbol{A}]^{-1}\Big]^T$$

Example 2

$$\phi(\boldsymbol{q}) = \Big[\frac{\beta}{\boldsymbol{q}^T[\boldsymbol{I}+\boldsymbol{A}+\boldsymbol{A}^2+\cdots+\boldsymbol{A}^{n-1}]\boldsymbol{l}}\boldsymbol{q}^T[\boldsymbol{I}+\boldsymbol{A}+\boldsymbol{A}^2+\cdots+\boldsymbol{A}^{n-1}]\Big]^T$$

Example 3 (Lippi)

$$\phi(\boldsymbol{q}) = \Big[\frac{\beta}{\boldsymbol{b}^T[\lambda(\boldsymbol{q})\boldsymbol{I}-\boldsymbol{A}]^{-1}\boldsymbol{l}}\boldsymbol{b}^T[\lambda(\boldsymbol{q})\boldsymbol{I}-\boldsymbol{A}]^{-1}\Big]^T$$

$\boldsymbol{b}$ is a given positive vector.

In this paper, we add a new $\phi(\boldsymbol{q})$ such that $\boldsymbol{x^n} = A\boldsymbol{x^{n-1}}$ each n. This modification is simpler than those in previous studies.

6 THE RANGE OF APPLICATION

6.1 Problem

We postulate that the matrix A is positive, and our proof depends on this. However, this postulate is unrealistic, because the input matrix generally contains zero elements. If A contains zero elements , we need to determine whether the corrected imaginary experiment is sustainable. Let us inspect a matrix, such as $\boldsymbol{A} \geq 0$[12]:

$$\boldsymbol{A} = \begin{pmatrix} 0.1 & 0 & 0.1 \\ 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 \end{pmatrix}.$$

Let us multiply, from the left, both sides of equation (2) by this matrix. We can not change the inequality $\leq$ to the inequality $<$ by this operation. For example, let us consider a vector such as $\boldsymbol{\gamma} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}'$, and $\boldsymbol{\delta} =$

11) Salvadori [7], p.256

12) $\boldsymbol{A} \geq \boldsymbol{B}$ means that $a_n \geqq b_n$ for all n and $a_n > b_n$ for some n.

$\begin{pmatrix} 1 & 4 & 3 \end{pmatrix}'$. This gives rise to $\gamma \leq \delta$. When we multiply both sides of this inequality by $\boldsymbol{A}$ from left, we get $\boldsymbol{A}\gamma \leq \boldsymbol{A}\delta$, not $\boldsymbol{A}\gamma < \boldsymbol{A}\delta$. This means that the algorithm stops at this point and the sequence does not converge to the lower limit. When a matrix contains zero elements, our proof may fail. We elaborate on this issue in the next section.

6.2 Consideration

6.2.1 primitive matrix

When we square the above matrix, we get a positive matrix:

$$\boldsymbol{A}^2 = \begin{pmatrix} 0.02 & 0.01 & 0.02 \\ 0.03 & 0.02 & 0.03 \\ 0.03 & 0.02 & 0.03 \end{pmatrix} > 0.$$

In the case of positive matrix, we can translate inequality(2) into inequality(4) and thus, the algorithm can be revived. Consequently, we get a monotonic decreasing sequence and can prove the convergence to standard ratio.

Meanwhile, Frobenius showed the next theorem.

Theorem (Frobenius)**.** $\boldsymbol{A} \geq 0$ is primitive if and only if $\boldsymbol{A}^m > 0$ for some $m > 0$[13)]

According to this theorem, even if a matrix contains zero elements, when our corrected imaginary experiment is primitive, it is sustainable.

6.2.2 imprimitive matrix

Next, we consider the following matrix:

$$\boldsymbol{A} = \begin{pmatrix} 0 & 0.1 \\ 0.5 & 0 \end{pmatrix}.$$

13) On this theorem and its proof, see Mayer [3] p.678, also see Nikaido [9] and Nikaido [10].

In this case, the convergence value of the sequence $\{\alpha^i\}$, that is a Frobnius eigenvalue is calculated as $r(A) = 0.22367$. Taking $\boldsymbol{x^0} = \begin{pmatrix} 1 & 2 \end{pmatrix}'$ and conducting the corrected imaginary experiment, we get $\alpha^1(\boldsymbol{x}^1) = 0.5$, $\alpha^2(\boldsymbol{x}^2) = 0.5$, $\alpha^3(\boldsymbol{x}^3) = 0.5$, $\alpha^4(\boldsymbol{x}^4) = 0.5$ ... , that is,

$$0.22367 = r(A) < \cdots \alpha^4(\boldsymbol{x}^4) = \alpha^3(\boldsymbol{x}^3) = \alpha^2(\boldsymbol{x}^2) = \alpha^1(\boldsymbol{x}^1) = 0.5.$$

We see that $\alpha^1 = \alpha^2 = \alpha^3 = \alpha^4 = \cdots$. These values are the limit value, which is not a Frobnius eigenvalue. The sequence is monotonic non-increasing but not monotonic decreasing. However, the sequence $\{\alpha^i\}$ does not converge to $r(A)$ and the proof fails.

The above matrix is a kind of imprimitive matrix. An imprimitive matrix has the following characteristic. Let us take any number ν and multiply $\boldsymbol{A}$ by ν; then, we get

$$\boldsymbol{A}^\nu = \begin{pmatrix} 0 & 0.1^\nu \\ 0.5^\nu & 0 \end{pmatrix}.$$

If a matrix is imprimitive, when raised to any numeral power, it contains zero elements. Consequently, we can not apply our corrected imaginary experiment in this case[14].

7 Conclusion and Discussion

In this paper, we investigated the imaginary experiment in section37 of Sraffa's *Production of Commodities by Means of Commodities.* Sraffa contends that "any actual economic system of the type we have been considering can always be transformed into a Standard system may be shown

14) Imprimitivity is a troublesome problem not only in our discussion but also in the Turnpike Theorem. Morishima says, "In proving the Turnpike Theorem above, we have assumed that the von Neumann activity set A_ϵ is *primitive*(or acyclic). It is clear from general economic considerations that this assumption is not a highly probable condition. For example, the two-sector system where the sole input of each sector is the output of the other sector is imprimitive." Morishima [6] p.171.

by an imaginary experiment". The imaginary experiment is an algorithm by which a calculation is repeated, in this study's terminology, from Step 2 to Step3 and then from Step3 to Step2, and so forth until the real economy can be transformed into a Standard System. Sraffa thought that this algorithm can work well without specializing $\boldsymbol{x}$. However, this is not true. Sraffa overlooked the fact that when the movement of $\boldsymbol{x}$ is not set properly, the algorithm does not always work. Our modification is to substitute $\boldsymbol{x^n} = A\boldsymbol{x^{n-1}}\,(i = 2, 3, \cdots)$. Making these corrections in his proof, Sraffa's intention, that the real economy can be transformed into Standard system, is almost accomplished. The revised algorithm is as follows, (see figure 3).

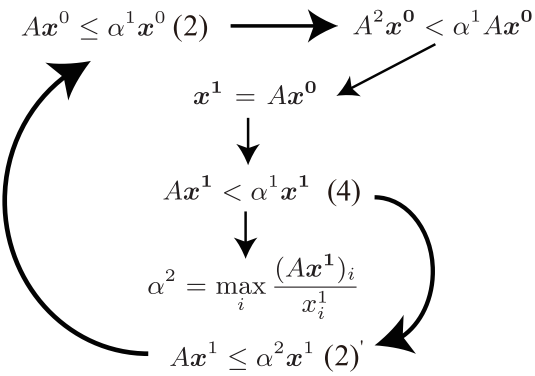


figure 3

However, when an input matrix is imprimitive, our correction results in failure.

8 Appendix

8.1 Appendix (Primary sources) : *Production of Commodities by Means of Commodities*, section 37

That any actual economic system of the type we have been considering can always be transformed into a Standard system may be shown by an imaginary experiment.

(The experiment involves two types of alternating steps. One type consists in changing the proportions of the industries; the other in reducing in the same ratio the quantities produced by all industries, while leaving unchanged the quantities used as means of production.)

We start by adjusting the proportions of the industries of the system in such a way that of each basic commodity a larger quantity is produced than is strictly necessary for replacement.

Let us next imagine gradually to reduce by means of successive small proportionate cuts the product of all the industries, without interfering with the quantities of labour and means of production that they employ.

As soon as the cuts reduce the production of any one commodity to the minimum level required for replacement, we readjust the proportions of the industries so that there should again be a surplus of each product (while keeping constant the quantity of labour employed in the aggregate). This is always feasible so long as there is a surplus of some commodities and a deficit of none.

We continue with such an alternation of proportionate cuts with the re-establishment of a surplus for each product until we reach the point where the products have been reduced to such an extent that all-round replacement is just possible without leaving anything as surplus product.

Since to reach this position the products of all the industries have been cut in the same proportion we are now able to restore the original conditions of production by increasing the quantity produced in each industry by a uniform rate; we do not, on the other hand, disturb the proportions to which the industries have been brought. The uniform rate which restores the original conditions of production is R and the proportions attained by the industries are the proportions of the Standard system.

8.2 Mathematical Appendix A

We show that $\{\alpha \mid \alpha \boldsymbol{x} > A\boldsymbol{x}\}$ is bounded from below by the smallest column of A. Let $S_n = \left\{\boldsymbol{x} \mid \boldsymbol{x} \geq 0, \sum_{j=1}^n x_j = 1\right\}$ be a standard simplex. Sequence $\alpha(\boldsymbol{x}) = \max_{x_i \neq 0} (A\boldsymbol{x})_i / x_i$ is homogeneous of degree 0 with x_i[15). Then we can take any $\boldsymbol{x}$ in $\boldsymbol{x} \in S_n$. From the definition of $\alpha(\boldsymbol{x})$, we have $\sum_{j=1}^n a_{ij} x_j \leq \alpha(\boldsymbol{x}) x_i$ $(i = 1, 2, \ldots, n)$. Noting $\sum_{i=1}^n x_i = 1$ and summing with respect i, we have

$$\sum_{i=1}^n \sum_{j=1}^n a_{ij} x_j \leq \sum_{i=1}^n \alpha(\boldsymbol{x}) x_i$$

$$= \alpha(\boldsymbol{x}).$$

Now, we let the column sums of a matrix A be

$$c_j = \sum_{i=1}^n a_{ij}, \quad j = 1, 2, \ldots, n. \quad ,$$

and the right side of the inequality is described as

15) For any $t > 0$ and $\boldsymbol{x} \in S_n$, we get

$$\alpha(t\boldsymbol{x}) = \max_{(tx)_i \neq 0} \frac{(A(t\boldsymbol{x}))_i}{(tx)_i} = \max_{(tx)_i \neq 0} \frac{t(A\boldsymbol{x})_i}{tx_i}$$

$$= \max_{x_i \neq 0} \frac{(A\boldsymbol{x})_i}{x_i} = \alpha(\boldsymbol{x}).$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_j = \sum_{j=1}^{n} x_j \sum_{i=1}^{n} a_{ij}$$
$$= \sum_{j=1}^{n} x_j c_j$$
$$\geq \min_j c_j .$$

Therefore,

$$\alpha(\boldsymbol{x}) \geq \min_j c_j . \qquad \blacksquare$$

8.3 Mathematical Appendix B

Suppose that $\epsilon = \min_{i,j} a_{ij}$ and $\boldsymbol{z} > 0$. Then, there exist the following inequalities:

$$\sum_{j=1}^{n} a_{ij} z_j \geq \sum_{j=1}^{n} \epsilon z_j \geq \epsilon \parallel \boldsymbol{z} \parallel_\infty .$$

If we apply these inequalities to the positive vector $\alpha_n \boldsymbol{x}^n - A\boldsymbol{x}^n$, we have

$$(7) \quad \sum_{j=1}^{n} a_{ij}(A\boldsymbol{x}^n - \alpha_n \boldsymbol{x}^n)_j \geq \epsilon \parallel \alpha_n \boldsymbol{x}^n - A\boldsymbol{x}^n \parallel_\infty \quad (i = 1, \ldots, n),$$

where $(\alpha_n \boldsymbol{x}^n - A\boldsymbol{x}^n)_j$ is the jth component of the vector $\alpha_n \boldsymbol{x}^n - A\boldsymbol{x}^n$. If we multiply $\dfrac{1}{\parallel A\boldsymbol{x}^n \parallel_\infty}$ both sides of (7) by the left-hand side , we see that

$$(8) \quad \frac{1}{\parallel A\boldsymbol{x}^n \parallel_\infty} \sum_{j=1}^{n} a_{ij}(\alpha_n \boldsymbol{x}^n - A\boldsymbol{x}^n)_j \geq \frac{1}{\parallel A\boldsymbol{x}^n \parallel_\infty} \epsilon \parallel \alpha_n \boldsymbol{x}^n - A\boldsymbol{x}^n \parallel_\infty \quad (i = 1, \ldots, n).$$

From the definition $\boldsymbol{x}^{n+1} = \dfrac{A\boldsymbol{x}^n}{\parallel A\boldsymbol{x}^n \parallel_\infty}$, the left-hand side of (8) is transformed as follows: for i=1,. . . ,n,

$$\frac{1}{\parallel A\boldsymbol{x}^n \parallel_\infty} \sum_{j=1}^{n} a_{ij}(A\boldsymbol{x}^n - \alpha_n \boldsymbol{x}^n)_j$$
$$= \frac{\alpha^n}{\parallel A\boldsymbol{x}^n \parallel_\infty} \sum_{j=1}^{n} a_{ij} \left(\boldsymbol{x}^n\right)_j - \frac{1}{\parallel A\boldsymbol{x}^n \parallel_\infty} \sum_{j=1}^{n} a_{ij} \left(A\boldsymbol{x}^n\right)_j$$
$$= \frac{\alpha^n}{\parallel A\boldsymbol{x}^n \parallel_\infty} \sum_{j=1}^{n} a_{ij} \left(\boldsymbol{x}^n\right)_j - \sum_{j=1}^{n} a_{ij} \left(\frac{A}{\parallel A\boldsymbol{x}^n\boldsymbol{x}^n \parallel_\infty}\right)_j$$
$$= \alpha^n \left(\boldsymbol{x}^{n+1}\right)_i - \sum_{j=1}^{n} a_{ij} \left(\boldsymbol{x}^{n+1}\right)_j \quad (i = 1, \ldots, n) .$$

Now, we define α_{n+1} as $\max\limits_{1\le i\le n} \dfrac{(A\boldsymbol{x}^{n+1})_i}{x_i^{n+1}}$ and let k be the component of the vector $\boldsymbol{x}^{n+1}$ that satisfies the definition of α_{n+1}. Then, we transform the left-hand side of (8) with respect to its kth component as follows:

$$\alpha^n \left(\boldsymbol{x}^{n+1}\right)_k - \sum_{j=1}^{n} a_{kj} \left(\boldsymbol{x}^{n+1}\right)_j = \alpha^{n+1} \left(\boldsymbol{x}^{n+1}\right)_k - \alpha^n \left(\boldsymbol{x}^{n+1}\right)_k$$

$$= \left(\alpha^n - \alpha^{n+1}\right) \left(\boldsymbol{x}^{n+1}\right)_k$$

Concerning the kth component of the inequality (8), we obtain

(9) $$\left(\alpha^n - \alpha^{n+1}\right) \left(\boldsymbol{x}^{n+1}\right)_k \ge \frac{\epsilon}{\| A\boldsymbol{x}^n \|_\infty} \| \alpha^n \boldsymbol{x}^n - A\boldsymbol{x}^n \|_\infty .$$

Since $\| \boldsymbol{x}^{n+1} \|_\infty \le 1$, this implies

(10) $$\frac{\| A\boldsymbol{x}^n \|_\infty}{\epsilon} \left(\alpha_n - \alpha_{n+1}\right) \ge \| \alpha_n \boldsymbol{x}^n - A\boldsymbol{x}^n \|_\infty .$$

$\lim\limits_{n\to\infty} \alpha_n = \alpha^*$ implies $\lim\limits_{n\to\infty} (\alpha^n - \alpha^{n+1}) = 0$, and therefore, this inequality indicates that

(11) $$\lim_{n\to\infty} \| \alpha_n \boldsymbol{x}^n - A\boldsymbol{x}^n \|_\infty = 0.$$

Let $Q = \left\{\boldsymbol{x} \in R^n \mid \ \| \boldsymbol{x} \|_\infty \le 1\right\}$. Because $\| \boldsymbol{x}^n \|_\infty \le 1 \ \ (i = 1, \ldots, n)$, then $\left\{\boldsymbol{x}^n\right\} \subset Q$. Q is a compact set, and thus, exists a converging subsequence of $\left\{\boldsymbol{x}^n\right\}$. Suppose it is $\left\{\boldsymbol{x}^{n_m}\right\}_{m=1,2,\ldots}$; then, we obtain

$$\lim_{m\to\infty} \boldsymbol{x}^{n_m} = \boldsymbol{x}^* \quad (\boldsymbol{x} \in Q).$$

Therefore,

$$\lim_{m\to\infty} \| \alpha^{n_m} \boldsymbol{x}^{n_m} - A\boldsymbol{x}^{n_m} \|_\infty = \| \alpha^* \boldsymbol{x}^* - A\boldsymbol{x}^* \|_\infty = 0.$$

By the definition of $\infty-$norm, if $\| \boldsymbol{z} \|_\infty = 0$ then $\boldsymbol{z} = 0$. We obtain

(12) $$A\boldsymbol{x}^* = \alpha^* \boldsymbol{x}^*$$

$\boldsymbol{x}^* > 0, \ \ \alpha^* > 0$ are obvious. ■

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