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Large Deviations for Regenerative Processes

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Abstract

In this paper, a large deviation principle for the long time behavior of certain regenerative processes is proved. The renewal theorem plays a crucial role in the proof.

1 Introduction

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space and $\xi_1, \xi_2, \ldots$ be the non-negative independent identically distributed (i.i.d.) random variables on Ω with the same probability law μ on $\mathbb{R}_+ = [0, \infty)$.In the present paper we assume that

$$\text{(A)} \quad \mathrm{Supp}(\mu) \neq \{0\} \quad \text{and} \quad \int_0^\infty e^{c \cdot t^{1+\varepsilon}} \mu(dt) < \infty \text{ for some } c > 0 \text{ and } \varepsilon > 0.$$

Under this assumption, the ξ_k's are integrable. Let $T_0, T_1, \ldots$ be the renewal times, defined by $T_0 = 0$, and $T_k = T_{k-1} + \xi_k$ for $k \geq 1$. We let $N_t = \inf\{k : T_k > t\}$, the number of renewals in $[0, t]$. We are interested in the long time asymptotic behavior of the 'residual waiting time' $R_t = T_{N_t} - t$. It is known (c.f. Chapter 3 of Durrett [**1**] and Chapter 4 of Bremaud [**2**]) that for all $x > 0$,

$$P(R_t > x) \rightarrow \tilde{\mu}([x, \infty))$$

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as $t \to \infty$. Here, $\tilde{\mu}(ds) = \frac{1}{z_\mu}\bar{F}(s)ds$ is a probability measure on $\mathbb{R}_+$ where $\bar{F}(s) = \mu([s, \infty))$ and $z_\mu = E[\xi_1] = \int_0^\infty s\mu(ds) = \int_0^\infty \bar{F}(s)ds$. In other words, the probability law of R_t converges weakly to $\tilde{\mu}$ in the space $\mathcal{M}_1(\mathbb{R}_+)$ of probability measures in $\mathbb{R}_+$ endowed with the weak topology. This result indicates the ergodic theorem for the empirical measure

$$L_t = \frac{1}{t}\int_0^t \delta_{R_s} ds.$$

of $\{R_t, t \geq 0\}$. Namely, the law of L_t converges to $\delta_{\tilde{\mu}}$ weakly as $t \to \infty$ in the space of probability measures on $\mathcal{M}_1(\mathbb{R}_+)$. The purpose of the present paper is to show the large deviation principle for $\{L_t, t \geq 0.\}$

For any $\nu \in \mathcal{M}_1(\mathbb{R}_+)$ for which $z_\nu = \int_{\mathbb{R}_+} \nu([s, \infty))ds < \infty$, we denote by $\tilde{\nu}$ another probability measure on $\mathbb{R}_+$ given by $\tilde{\nu}(ds) = \frac{\nu([s,\infty))}{z_\nu} ds$. Let $\tilde{\mathcal{M}}_1(\mathbb{R}_+) = \{\tilde{\nu};\ \nu \in \mathcal{M}_1(\mathbb{R}_+)\}$ and let

$$\begin{aligned} \mathcal{N}(\mathbb{R}_+) &= \{\tilde{\nu} \in \tilde{\mathcal{M}}_1(\mathbb{R}_+);\ \nu \text{ is absolutely continuous with respect to } \mu \\ &\quad \text{and } \log \tfrac{d\nu}{d\mu} \in L^1(\nu)\}. \end{aligned}$$

Our main result is as follows.

Theorem 1. *Let $\lambda = \lambda(\phi) \in \mathbb{R}$ be given by the formula*

$$\int_{\mathbb{R}_+} \exp\left(\int_0^t \phi(s)ds - \lambda t\right)\mu(dt) = 1$$

for any $\phi \in C_b(\mathbb{R}_+ \to \mathbb{R})$ and let I on $\mathcal{M}_1(\mathbb{R}_+)$ be defined by

$$I(m) = \sup_{\phi \in C_b(\mathbb{R}_+)} \left\{\int_{\mathbb{R}_+} \phi(t)m(dt) - \lambda(\phi)\right\}.$$

Then I is a convex rate function on $\mathcal{M}_1(\mathbb{R}_+)$ for which the following holds:

1.

$$I(m) = \begin{cases} \dfrac{1}{z_\nu} \displaystyle\int_{\mathbb{R}_+} \log \frac{d\nu}{d\mu}(t) d\nu(t) & \textit{if } m = \tilde{\nu} \textit{ for some } \tilde{\nu} \in \mathcal{N}(\mathbb{R}_+) \\ \\ \infty & \textit{otherwise} \end{cases}$$

2. $I(m) = 0$ if and only if $m = \tilde{\mu}$.

3. The large deviation for empirical measure of $\{R_t\}$ holds with the rate function I, namely, for any open set G in $\mathcal{M}(\mathbb{R}_+)$,

$$\limsup_{t \to \infty} \frac{1}{t} \log P(L_t \in G) \geq -\inf\{I(\nu), \ \nu \in G\}$$

and for any closed set F in $\mathcal{M}(\mathbb{R}_+)$,

$$\liminf_{t \to \infty} \frac{1}{t} \log P(L_t \in F) \leq -\inf\{I(\nu), \ \nu \in F\}.$$

Let us give some remarks. The large deviation principle for the empirical distribution of the i.i.d. random variables and Markov processes (both in discrete as well as continuous time) with some mixing condition is well-established, see for example, Deuschel & Stroock [**3**]. If there are regeneration points, or the renewal times of the process, then the states between these regeneration points constitute independent random variables with the same distribution, and thus it seems likely that the long time empirical measure of the process satisfies the large deviation property as the one of i.i.d. random variables does. However, this result shows that the rate function for the regenerative processes is different from the one of the i.i.d. case.

In terms of the method adopted in this paper, we make good use of the renewal equation and the renewal theorem to examine the large deviation principle. This strategy has yet to be fully exploited, and should be further investigated for other applications.

The paper is organized as follows. In section 2 we state two results on the renewal theory without proofs. The first one is the direct conclusion of the so-called renewal theorem, and the second one is the corollary of it. We apply both statements in the following sections. In section 3, by using the first result of section 2, we establish the variational formula for the logarithmic moment generating function of the law of *Rt*. Using this, as usual, we define the rate function *I* as the Legendre transform of the moment generating function, and obtain the first part of our Theorem. In section 4, we prove the upper bound. To show the exponential tightness, we use the same argument based on the renewal theorem as in section 3. In section 5, we prove the lower bound, completing the proof of the theorem. Again, we apply the result from the renewal theorem.

2 Renewal Theorem

We first review the well-known renewal theorem, since it plays the crucial role in the proof of the main theorem. Let F be a probability distribution function and h be a given function. Let H be the solution of the renewal equation. Then, we have the following so-called renewal theorem(c.f. Chapter 3 of Durrett[**1**]).

Proposition 1. *If μ is non-arithmetic and h is directly Riemann integrable, then, for the solution H of the following renewal equation*

$$H(t) = h(t) + \int_0^t H(t-s)\mu(ds),$$

we have

$$H(t) \to \frac{1}{z_\mu}\int_0^\infty h(s)ds$$

as $t \to \infty$.

Here, we remark that if $h(t) \geq 0$ is decreasing with $h(0) < \infty$ and $\int_0^\infty h(t)dt < \infty$, then h is directly integrable. As a corollary of this theorem, we obtain another version of the renewal theorem, which we will use later.

Proposition 2. *Suppose that μ is non-arithmetic, h(t) converges to a constant $\bar{h}$ as $t \to \infty$ and $h(t)-\bar{h}$ is directly integrable. Then, for the solution H of the following renewal equation*

$$H(t) = h(t) + \int_0^t H(t-s)\mu(ds),$$

we have

$$\frac{H(t)}{t} \to \frac{h}{z_\mu}.$$

as t $\to \infty$.

3 Rate Function

By the assumption (**A**), for any $\phi \in C_b(\mathbb{R}_+ \to \mathbb{R})$, there is a $\lambda = \lambda(\phi) \in \mathbb{R}$ given by the formula

$$\int_{\mathbb{R}_+} \exp\left(\int_0^t \phi(s)ds - \lambda t\right) \mu(dt) = 1. \tag{1}$$

For this we have the following.

Proposition 3. *For any* $\phi \in C_b(\mathbb{R}_+ \to \mathbb{R})$, *we have*

$$\lim_{t\to\infty} \frac{1}{t} \log E\Big[\exp(\int_0^t \phi(R_u)du)\Big] = \lambda(\phi), \tag{2}$$

where $\lambda(\phi)$ *is efined by* (*1*).

Proof. Due to (1), for any $\phi \in C_b(\mathbb{R}_+ \to \mathbb{R})$, $\exp\left(\int_0^t \phi(s)ds - \lambda t\right) \mu(dt)$ is a probability measure on $\mathbb{R}_+$, which we will denote by $\mu_\phi(dt)$. By Proposition **1**, for the solution H of the renewal equation

$$H(t) = h(t) + \int_0^t H(t-s)\exp\left(\int_0^s \phi(u)du - \lambda s\right) \mu(ds),$$

we have

$$H(t) \to \frac{1}{z_\phi} \int_0^\infty h(s)ds. \tag{3}$$

Here, z_ϕ is the mean with respect to $\mu_\phi(dt)$, or

$$z_\phi = \int_0^\infty s \exp\left(\int_0^s \phi(u)du - \lambda s\right) \mu(ds).$$

Since $R_u = u$ for all $u < t = \xi_1$, we see that

$$E\Big[\exp(\int_0^t \phi(R_u)du)\Big]$$

$$(4) \qquad = E\Big[\exp(\int_0^t \phi(R_u)du); t < \xi_1\Big] + \int_0^t E\Big[\exp(\int_0^t \phi(R_u)du)|\xi_1 = s\Big]\mu(ds)$$

$$= \exp(\int_0^t \phi(u)du)\bar{F}(t) + \int_0^t E\Big[\exp(\int_0^t \phi(R_u)du)|\xi_1 = s\Big]\mu(ds),$$

where $\bar{F}(s) = \mu([s, \infty))$, and we also observe that

$$E\Big[\exp(\int_0^t \phi(R_u)du)|\xi_1 = s\Big] = \exp(\int_0^s \phi(u)du)E\Big[\exp(\int_s^t \phi(R_u)du)|\xi_1 = s\Big]$$

$$= \exp(\int_0^s \phi(u)du)E\Big[\exp(\int_0^{t-s} \phi(R_u)du)\Big]$$

by the regenerative nature of R_t. Hence, we see that $U(t) = E\Big[\exp(\int_0^t \phi(R_u)du)\Big]$ satisfies the relation

$$(5) \qquad U(t) = \exp(\int_0^t \phi(u)du)\bar{F}(t) + \int_0^t U(t-s)\exp(\int_0^s \phi(u)du)\mu(ds)$$

Now we take the $\lambda = \lambda(\phi)$ which satisfies (1) and multiply both sides of (5) by $e^{-\lambda t}$ to obtain

$$e^{-\lambda t}U(t) = \exp(\int_0^t \phi(u)du - \lambda t)\bar{F}(t) + \int_0^t e^{-\lambda(t-s)}U(t-s)\exp(\int_0^s \phi(u)du - \lambda s)\mu(ds).$$

Hence from, we see that

$$(6) \qquad e^{-\lambda t}U(t) \to \frac{1}{z_\phi}\int_0^\infty \exp(\int_0^s \phi(u)du - \lambda s)\bar{F}(s)ds,$$

where the RHS of (6) exists under our assumption (**A**). This implies our assertion. □

Now let I be a functional on $\mathcal{M}_1(\mathbb{R}_+)$ defined by

(7) $$I(m) = \sup_{\phi \in C_b(\mathbb{R}_+)} \left\{ \int_{\mathbb{R}_+} \phi(t) m(dt) - \lambda(\phi) \right\}.$$

We propose the following, which constitutes the first part of the main theorem.

Proposition 4. *I is a convex rate function and*

$$I(\tilde{\nu}) = \frac{1}{z_\nu} \int_{\mathbb{R}_+} \log \frac{d\nu}{d\mu}(t) d\nu(t),$$

for $\tilde{\nu} \in \mathcal{N}(\mathbb{R}_+)$ *and* $I(m) = \infty$ *for any* $m \in \mathcal{M}_1(\mathbb{R}_+) \setminus \mathcal{N}(\mathbb{R}_+)$.

Proof. Since I is the supremum of continuous linear functionals, it is lower semi-continous and convex. Now fo any $M > 0$, let $\alpha(M)$ be given by

$$\int_{[\alpha(M),\infty)} e^{Mt} \mu(dt) \leq 1 - \int_{\mathbb{R}_+} e^{-t} \mu(dt).$$

Then for all $\phi \in C_b(\mathbb{R}_+ \to \mathbb{R})$ satisfying the conditon
(♯) $\phi(t) \in [0, M]$ for all $t \in \mathbb{R}_+$ and $\phi(t) = 0$ for all $t \leq \alpha(M)$,
$\lambda = \lambda(\phi)$ given by (1) is non-negative and so

$$\int_{[\alpha(M),\infty)} \exp\left(\int_0^t \phi(s) ds - \lambda t \right) \mu(dt) \leq 1 - \int_{\mathbb{R}_+} e^{-t} \mu(dt).$$

Thus, by (1),

$$\begin{aligned}
\int_{\mathbb{R}_+} e^{-\lambda t} \mu(dt) &\geq \int_{[0,\alpha(M))} e^{-\lambda t} \mu(dt) \\
&= \int_{[0,\alpha(M))} \exp\Big(\int_0^t \phi(s) ds - \lambda t \Big) \mu(dt) \geq \int_{\mathbb{R}_+} e^{-t} \mu(dt),
\end{aligned}$$

which implies $\lambda(\phi) \leq 1$. Therefore, if $I(m) \leq L$, then for all ϕ satisfying (♯),

$$\int_{\mathbb{R}_+} \phi(t) m(dt) \leq I(m) + \lambda(\phi) \leq L + 1$$

and so $m([\alpha(M), \infty)) \leq \frac{L+1}{M}$. This implies that $\{m \in \mathcal{M}_1(\mathbb{R}_+); I(m) \leq L\}$ is compact in

$\mathcal{M}_1(\mathbb{R}_+)$.

Next, suppose that $\tilde{\nu} \in \mathcal{N}(\mathbb{R}_+)$. Let $\psi(t) = \frac{d\nu}{d\mu}(t)$, then from (1) we see that for all $\phi \in C_b(\mathbb{R}_+ \to \mathbb{R})$,

$$\int_{\mathbb{R}_+} \exp\left(\int_0^t \phi(s)ds - \lambda(\phi)t\right) \psi(t)^{-1}\nu(dt) = 1$$

and so by Jensen's inequality, we get

$$\int_{\mathbb{R}_+} \left(\int_0^t \phi(s)ds - \lambda(\phi)t - \log\psi(t)\right) \nu(dt) \leq 0$$

or

$$\int_{\mathbb{R}_+} \left(\int_0^t \phi(s)ds\right) \nu(dt) - \left(\int_{\mathbb{R}_+} t\nu(dt)\right) \lambda(\phi) \leq \int_{\mathbb{R}_+} \log\psi(t)\nu(dt).$$

Noting that $z_\nu < \infty$, by the integration by parts, we obtain

$$\int_{\mathbb{R}_+} \phi(t)\nu([t,\infty))dt - z_\nu\lambda(\phi) \leq \int_{\mathbb{R}_+} \log\psi(t)\nu(dt)$$

and so by dividing by z_ν,

$$\int_{\mathbb{R}_+} \phi(t)\tilde{\nu}(dt) - \lambda(\phi) \leq \frac{1}{z_\nu}\int_{\mathbb{R}_+} \log\psi(t)\nu(dt).$$

Taking the supremum over ϕ, we see that

$$I(\tilde{\nu}) \leq \frac{1}{z_\nu}\int_{\mathbb{R}_+} \log\psi(t)\nu(dt).$$

It remains to show that if $I(m) < \infty$, then $m = \tilde{\nu}$ for some $\nu \in \mathcal{N}(\mathbb{R}_+)$ and $I(\tilde{\nu}) \geq \frac{1}{z_\nu}\int_{\mathbb{R}_+} \log\psi(t)\nu(dt)$.

In the case there is $\phi \in C_b(\mathbb{R}_+ \to \mathbb{R})$ such that

$$\int_0^t \phi(s)ds = \log\frac{d\nu}{d\mu}(t)$$

for all $t \in \mathbb{R}_+$, we see that

$$\int_{\mathbb{R}_+} \exp\left(\int_0^t \phi(s)ds\right) \mu(dt) = \int_{\mathbb{R}_+} 1\nu(dt) = 1$$

and so $\lambda(\phi)=0$. Hence

$$\begin{aligned} I(\tilde{\nu}) &\geq \int_{\mathbb{R}_+} \phi\tilde{\nu}(dt) = \frac{1}{z_\nu}\int_0^\infty \phi(t)\nu([t,\infty))dt \\ &= \frac{1}{z_\nu}\int_{\mathbb{R}_+}\left(\int_0^t \phi(s)ds\right)\nu(dt) = \frac{1}{z_\nu}\int_0^\infty \log\frac{d\nu}{d\mu}(t)\nu(dt). \end{aligned}$$

By the standard truncation argument we can draw the same conclusion on the other cases, and thus the proposition follows. □

4 Proof of Upper Bounds

In this section we prove the upper large deviation bounds for R_t. First, we prepare two lemmas.

Lemma 1. *Let F be a compact set in $\mathcal{M}_1(\mathbb{R}_+)$. Then*

$$\limsup_{t\to\infty}\frac{1}{t}\log P(L_t \in F) \leq -\inf\{I(\nu),\ \nu \in F\}.$$

Proof. Let $\ell = \inf\{I(\nu), \nu \in F\}$. For arbitrary $\varepsilon > 0$ and each $\nu \in F$, we can choose a $\phi_\nu \in C_b(\mathbb{R}_+)$ so that $\int_{\mathbb{R}_+}\phi_\nu(t)\nu(dt) - \lambda(\phi_\nu) \geq \ell - \varepsilon$. Next, for each $\nu \in F$, we choose an open neighborhood B_ν of ν in $\mathcal{M}_1(\mathbb{R}_+)$ so that

$$\sup_{m\in B_\nu}\left|\int_{\mathbb{R}_+}\phi_\nu dm - \int_{\mathbb{R}_+}\phi_\nu d\nu\right| < \varepsilon.$$

Since F is compact, we can choose $\nu_1, \cdots, \nu_N \in F$ so that $F \subset \bigcup_{i=1}^N B_{\nu_i}$. Clearly,

$$\limsup_{t\to\infty}\frac{1}{t}\log P(L_t \in F) \leq \limsup_{t\to\infty}\frac{1}{t}\log\sup_i P(R_t \in B_{\nu_i}).$$

But for $\nu \in F$,

$$\begin{aligned} P(R_t \in B_{\nu_i}) &\leq E\left[\exp\left(\int_0^t \phi_\nu(R_s)ds\right), R_t \in B_\nu\right] \cdot \sup_{m\in B_\nu} \exp(-t\int_{[0,t]} \phi_\nu dm) \\ &\leq E\left[\exp\left(\int_0^t \phi_\nu(R_s)ds\right)\right] \cdot \exp\Big(-t(\lambda(\phi_\nu) + \ell - 2\varepsilon)\Big), \end{aligned}$$

and thus, by (2),

$$\limsup_{t\to\infty} \frac{1}{t} \log P(L_t \in F) \leq -\ell + 2\varepsilon.$$

As ε is arbitrary, this implies our assertion. □

Lemma 2. *For each $M > 0$, there exists a compact set $C(M)$ in $\mathcal{M}_1(\mathbb{R}_+)$ such that*

$$\limsup_{t\to\infty} \frac{1}{t} \log P(L_t \in C(M)^c) \leq M,$$

where $C(M)^c = \mathcal{M}_1(\mathbb{R}_+) \setminus C(M)$.

Proof. By the assumption **(A)**, we can pick a $\phi \in C(\mathbb{R}_+ \to \mathbb{R})$ such that

- $\phi \geq 0$ and ϕ is non-decreasing on $\mathbb{R}_+$.
- $\{x \in \mathbb{R}_+;\ \phi(x) \leq M\}$ is compact in $\mathbb{R}_+$ for each $M \geq 0$.
- There is a $\lambda = \lambda(\phi)$ for which (1) holds.

The whole argument to derive the variational formula (2) still holds for this ϕ, and thus we have

$$\lim_{t\to\infty} \frac{1}{t} \log E\left[\exp(\int_0^t \phi(R_u)du)\right] = \lambda(\phi),$$

and so for sufficiently large t,

$$E\left[\exp(\int_0^t \phi(R_u)du)\right] \leq e^{(\lambda+1)t}. \tag{8}$$

Now set $K(M) = \{x \in \mathbb{R}_+; \phi(x) \leq M^2\}$. If $L_t(K(M)^c) \geq M^{-1}$, then

$\frac{1}{t}|s \in [0,t];\ \phi(R_s) > M^2| \geq M^{-1}$, and so

$$\frac{1}{t}\int_0^t \phi(R_s)ds \geq M^2 \cdot \frac{1}{t}|s \in [0,t];\ \phi(R_s) > M^2| \geq M.$$

Hence, from (8),

$$\begin{aligned} P(L_t(K(M)) \geq M^{-1}) &\leq P\left(\frac{1}{t}\int_0^t \phi(R_s)ds \geq M\right) \\ &\leq E\left[\exp(\int_0^t \phi(R_s)ds)\right] e^{-Mt} \leq e^{-(M-\lambda-1)t}. \end{aligned}$$

Hence, if we let $C(M) = \bigcap_{k=0}^{\infty}\left\{\nu \in \mathcal{M}_1(\mathbb{R}_+);\ \nu(K(M+\lambda+k+1)^c) \leq \frac{1}{M+\lambda+k+1}\right\}$, then $C(M)$ is compact in $\mathcal{M}_1(\mathbb{R}_+)$ and so our assertion follows from

$$\begin{aligned} P(R_t \in C(M)^c) &\leq \sum_{k=1}^{\infty} P\Big(R_t(K(M+\lambda+k+1)^c) \leq \frac{1}{M+\lambda+k+1}\Big) \\ &\leq \sum_{k=0}^{\infty} e^{-(M+k)t} \leq e^{-Mt}. \end{aligned}$$

□

Proposition 5. *For any open set F in $\mathcal{M}(\mathbb{R}_+)$,*

$$\limsup_{t\to\infty} \frac{1}{t}\log P(L_t \in F) \leq -\inf\{I(\nu),\ \nu \in F\}. \tag{9}$$

Proof. Since $F \subset (F \cap C(M)) \cup C(M)^c$, and $F \cap C(M)$ is compact for each M, by the preceding two lemmas,

$$\limsup_{t\to\infty} \frac{1}{t}\log P(L_t \in F) \leq \max\Big\{-\inf_{\nu \in F\cap C(M)} I(\nu), -M\Big\}.$$

□

Since M is arbitrary, by letting $M \to \infty$, we get (9).

5 Proof of Lower Bounds

In this section, we prove the lower large deviation bounds for L_t, completing the proof of the main theorem.

Proposition 6. *For any open set G in $\mathcal{M}_1(\mathbb{R}_+)$,*

$$\limsup_{t\to\infty} \frac{1}{t} \log P(L_t \in G) \geq -\inf\{I(\nu),\ \nu \in G\}. \tag{10}$$

Proof. It suffices to show that for any $\tilde{\nu} \in \mathcal{N}(\mathbb{R}_+)$ and for any $\varepsilon > 0$,

$$\liminf_{t\to\infty} \frac{1}{t} \log P(L_t \in B(\tilde{\nu}, \varepsilon)) \geq -I(\tilde{\nu}). \tag{11}$$

Here, $B(\tilde{\nu}, \varepsilon)$ is an open ball in $\mathcal{M}_1(\mathbb{R}_+)$. We denote by $P_t \in \mathcal{M}_1(D([0, t] \to \mathbb{R}_+))$ the law of the stochastic process $\{R_s, s \in [0, t])\}$, whereas let $Q_t \in \mathcal{M}_1(D([0, t] \to \mathbb{R}_+))$ be the law of the stochastic process $\{R_s, s \in [0, t]\}$ where the residual waiting time is given by the i.i.d. sequence ξ_k, $k = 1, 2,\ldots$ with the law ν. More precisely, $T_0, T_1,\ldots$ are the renewal times defined by $T_0 = 0$, and $T_k = T_{k-1} + \xi_k$ for $k \geq 1$, where the law of ξ_k's is ν. R_t is defined by $R_t = T_{N_t} - t$ where $N_t = \inf\{k : T_k > t\}$. Let $A_t = \{L_t \in B(\nu, \varepsilon)\}$, then by the ergodic theorem on $\{R.\}$ noted in the introduction, $Q_t(A_t) \to 1$ as $t \to \infty$. Thus, by Jensen's inequality, if we $Ft = \dfrac{dQ_t}{dP_t}$,

$$\begin{aligned}
\liminf_{t\to\infty} \frac{1}{t} \log P(A_t) &= \liminf_{t\to\infty} \frac{1}{t} \log E^{Q_t}[F_t^{-1}|A_t]Q(A_t) = \liminf_{t\to\infty} \frac{1}{t} \log E^{Q_t}[F_t^{-1}|A_t] \\
&\geq -\liminf_{t\to\infty} \frac{1}{t} E^{Q_t}[\log F_t|A_t] = -\liminf_{t\to\infty} \frac{1}{tQ(A_t)} \int_{A_t} \log F_t dQ_t \qquad (12)\\
&= -\liminf_{t\to\infty} \frac{1}{t} \int_{D([0,t]\to\mathbb{R}_+)} \log F_t dQ_t.
\end{aligned}$$

Here, we used the fact that $x \to x \log x$ is bounded below by e^{-1}. Let $H(t) = E^{Q^t}[\log F_t]$, By the regenerative nature of the process R_t, we see that

$$F_t = \frac{d\nu}{d\mu}(s) \cdot \frac{dQ_{t-s}}{dP_{t-s}}(\theta_s \cdot),$$

on the event $\{\xi_1 = s\}$ for some $s \leq t$, we see that

$$\begin{aligned} H(t) &= E^{Q_t}[\ \log F_t,\ \xi_1 > t] + \int_{[0,t]} E^{Q_t}[\ \log F_t |\ \xi_1 = s]\nu(ds) \\ &= E^{Q_t}[\ \log F_t,\ \xi_1 > t] + \int_{[0,t]} \log \frac{d\nu}{d\mu}(s)\nu(ds) + \int_{[0,t]} H(t-s)\nu(ds). \end{aligned}$$

Thus we get another renewal equation. The first term of RHS is bounded by $e^{-1}\nu(\xi_1 > t)$, and it goes to 0 as $t \to \infty$. Hence, applying Proposition **2** , we see that

$$\lim_{t\to\infty} \frac{H(t)}{t} \to \frac{1}{z_\nu} \int_{[0,\infty]} \log \frac{d\nu}{d\mu}(s)\nu(ds). \tag{13}$$

By (12) and (13), we see for any $\tilde{\nu} \in \mathcal{N}_\infty(\mathbb{R}_+)$

$$\liminf_{t\to\infty} \frac{1}{t} \log P(L_t \in B(\tilde{\nu}, \varepsilon)) \geq \frac{1}{z_\nu} \int_{[0,\infty]} \log \frac{d\nu}{d\mu}(s)\nu(ds),$$

and this implies our assertion. □

Reference

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