Algebraic combinatorics on perfectly matchable subgraph polytopes

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by

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Preface

The perfectly matchable subgraph polytope of a graph is a (0,1)-polytope associated with the vertex sets of matchings in the graph. In this thesis, we study algebraic properties (compressedness, Gorensteinness) of the toric rings of perfectly matchable subgraph polytopes based on [25]. In particular, we give a complete characterization of a graph whose perfectly matchable subgraph polytope is compressed.

The present thesis is organized as follows. In Chapter 1, we recall the definition of Gröbner basis based on [5]. Next we then confirm the basic terms of graph theory throughout [7] in Chapter 2. Moreover, in Chapter 3, we study convex polytopes from [14, 15]. We check the main theorem of present thesis in Chapter 4. In Chapter 5, we introduce relationships between $\mathscr{P}_G$ and other polytopes. In Chapter 6, in order to prove the main theorems, we examine inequalities which are facet-inducing for $\mathscr{P}_G$. In Chapter 7, we give a proof for Theorem 4.6. In Chapter 8, we give a proof for Theorems 4.7 and 4.8. Finally we write source codes for computing polytopes we studied and challenges for the future as appendix.

Acknowledgment

Without the support of many people and advice from my supervisor Professor Hidefumi Ohsugi, I could not have completed this work.
I wish to express deep appreciation to my supervisor Professor Hidefumi Ohsugi for his helpful advice and constant encouragement through this study.
Also, I would like to express my gratitude to Professor Akiyoshi Tsuchiya.

Contents

Chapter 1

Gröbner basis

At first, we recall a basic definitions of commutative algebra based on [5].

1.1 Polynomials

We start by definition of monomials.

Definition 1.1. A *monomials* in $x_1, \ldots, x_n$ is a product of the form

$$x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdots x_n^{\alpha_n},$$

where all of the exponents $\alpha_1, \ldots \alpha_n$ are nonnegative integers. The *total degree* of this monomial is the sum $\alpha_1 + \cdots + \alpha_n$.

We can simplify the notation for monomials as follows: let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be an n-tuple of nonnegative integers. Then we set

$$x^\alpha = x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdots x_n^{\alpha_n}.$$

When $\alpha = (0, \ldots, 0)$, note that $x^\alpha = 1$. We also let $|\alpha| = \alpha_1 + \cdots + \alpha_n$ denote the total degree of the monomial x^α.

Definition 1.2. A *polynomial* f in $x_1, \ldots, x_n$ with coefficients in a field K is a finite linear combination (with coefficients in K) of monomials. We will write a polynomial f in the form

$$f = \sum_\alpha a_\alpha x^\alpha, a_\alpha \in K,$$

where the sum is over a finite number of n-tuples $\alpha = (\alpha_1, \ldots, \alpha_n)$. The set of all polynomials in $x_1, \ldots, x_n$ with coefficients in K is denoted $K[x_1, \ldots, x_n]$.

When dealing with polynomial in a small number of variables, we will usually dispense with subscripts. Thus, polynomials in one, two and three variables lie in $K[x], K[x,y]$, and $K[x,y,z]$, respectively. For example,

$$f = 3x^2y^2z + \frac{11}{13}y^2z^3 - 100x^2yz + \frac{2}{3}y$$

is a polynomial in $\mathbb{Q}[x,y,z]$. We will usually use the letters f,g,h,p,q,r to refer to polynomials.

We will use the following terminology in dealing with polynomials.

Definition 1.3. Let $f = \sum_\alpha a_\alpha x^\alpha$ be the polynomial in $K[x_1,\dots,x_n]$.

(i) We call a_α the *coefficient* of the monomial x^α.

(ii) If $a_\alpha \neq 0$, then we call $a_\alpha x^\alpha$ a term of f.

(iii) The *total degree* of $f \neq 0$, denoted $\deg(f)$, is the maximum $|\alpha|$ such that the coefficient a_α is nonzero. The total degree of the zero polynomial is undefined.

For example, the polynomial $f = 3x^2y^2z + \frac{11}{13}y^2z^3 - 100x^2yz + \frac{2}{3}y$ has four terms and total degree five.

1.2 Ideals

We next define the basic algebraic objects.

Definition 1.4. A subset $I \subseteq K[x_1,\dots,x_n]$ is an *ideal* if it satisfies:

(i) $0 \in I$,

(ii) If $f,g \in I$, then $f+g \in I$,

(iii) If $f \in I$ and $h \in K[x_1,\dots,x_n]$, then $hf \in I$.

The first natural example of an ideal is the ideal generated by a finite number of polynomials.

Definition 1.5. Let $f_1,\dots,f_s$ be polynomials in $K[x_1,\dots,x_n]$. Then we set

$$\langle f_1,\dots,f_s\rangle = \left\{ \sum_{i=1}^{s} h_i f_i : h_1,\dots h_s \in K[x_1,\dots,x_n] \right\}$$

It is known that $\langle f_1,\dots,f_s\rangle$ is an ideal.

Lemma 1.6. *If $f_1,\dots,f_s \in K[x_1,\dots,x_n]$, then $\langle f_1,\dots,f_s\rangle$ is an ideal of $K[x_1,\dots,x_n]$. We will call $\langle f_1,\dots,f_s\rangle$ the ideal generated by $f_1,\dots,f_s$.*

We say that an ideal I is *finitely generated* if there exist $f_1,\dots,f_s \in K[x_1,\dots,x_n]$ such that $I = \langle f_1,\dots,f_s\rangle$, and we say that $f_1,\dots,f_s$, are a *bases* of I. Note that a given ideal may have many different bases. In this chapter, we will study one can choose an especially useful type of basis, called a Gröbner basis.

1.3 Polynomials of one variable

In this section, we will discuss polynomials of one variable and study the *division algorithm*.

We begin by discussing the division algorithm for polynomials in $K[x]$. At first, we should define a leading term of a polynomial in one variable.

Definition 1.7. Given a nonzero polynomial $f \in K[x]$, let

$$f = c_0x^m + c_1x^{m-1} + \cdots + c_m,$$

where $c_i \in K$ and $c_0 \neq 0$ [thus, $m = \deg(f)$]. Then we say that c_0x^m is the *leading term* of f, written $\mathrm{LT}(f) = c_0x^m$.

For example, if $f = 5x^2 - 7x + 10$, then $\mathrm{LT}(f) = 5x^2$. Notice also that if f and g are nonzero polynomials, then

$$\deg(f) \leq \deg(g) \Longleftrightarrow \mathrm{LT}(f) \text{ divides } \mathrm{LT}(g).$$

We can now describe the division algorithm.

Proposition 1.8. ***(The Division Algorithm)*** *Let K be a field and let g be a nonzero polynomial in $K[x]$. Then every $f \in K[x]$ can be written as*

$$f = qg + r,$$

where $q, r \in K[x]$, and either $r = 0$ or $\deg(r) < \deg(g)$. Furthermore, q and r are unique, and there is an algorithm for finding q and r.

A useful corollary of the division algorithm concerns the number of roots of a polynomial in one variable.

Corollary 1.9. *If K is a field and $f \in K[x]$ is a nonzero polynomial, then f has at most $\deg(f)$ roots in K.*

By Proposition 1.8, we can determine the structure of all ideals of $K[x]$.

Corollary 1.10. *If K is a field, then every ideal $K[x]$ can be written as $\langle f \rangle$ for some $f \in K[x]$. Furthermore, f is unique up to multiplication by a nonzero constant in K.*

Next, we define the greatest common divisor of polynomials.

Definition 1.11. A *greatest common divisor* of polynomials $f, g \in K[x]$ is a polynoial h such that:

(i) h divides f and g.

(ii) If p is another polynomial which divides f and g, then p divides h. When h has these properties, we write $h = \gcd(f, g)$.

Here are the main properties of gcd's.

Proposition 1.12. *Let $f, g \in K[x]$. Then:*

(i) $\gcd(f,g)$ *exists and is unique up to multiplication by a nonzero constant in K.*

(ii) $\gcd(f,g)$ *is a generator of the ideal* $\langle f,g\rangle$.

(iii) There is an algorithm for finding $\gcd(f,g)$.

Definition 1.13. A *greatest common divisor* of polynomials $f_1,\ldots,f_s \in K[x]$ is a polynomial h such that:

(i) h divides $f_1,\ldots,f_s$.

(ii) If p is another polynomial which divides $f_1,\ldots,f_s$ then p divides h.

When h has these properties, we write $h = \gcd(f_1,\ldots,f_s)$.

Here are main properties of these gcd's.

Proposition 1.14. *Let $f_1,\ldots,f_s \in K[x]$, where $s \geq 2$. Then:*

(i) $\gcd(f_1,\ldots,f_s)$ *exists and is unique up to multiplication by a nonzero constant in K.*

(ii) $\gcd(f_1,\ldots,f_s)$ *is a generator of the ideal* $\langle f_1,\ldots,f_s\rangle$.

(iii) If $s \geq 3$, then $\gcd(f_1,\ldots,f_s) = \gcd(f_1,\gcd(f_2,\ldots,f_s))$.

(iv) There is an algorithm for finding $\gcd(f_1,\ldots,f_s)$.

1.4 Ordering on the monomials in $K[x_1,\ldots,x_n]$

In this section, we will study how to *order* the terms of a polynomial.

Definition 1.15. A *monomial ordering* $>$ on $K[x_1,\ldots,x_n]$ is a relation $>$ on $\mathbb{Z}^n_{\geq 0}$, or equivalently, a relation on the set of monomials x^α, $\alpha \in \mathbb{Z}^n_{\geq 0}$, satisfying:

(i) $>$ is a total (or linear) ordering on $\mathbb{Z}^n_{\geq 0}$

(ii) If $\alpha > \beta$ and $\gamma \in \mathbb{Z}^n_{\geq 0}$, then $\alpha+\gamma > \beta+\gamma$

(iii) $>$ is a well-ordering on $\mathbb{Z}^n_{\geq 0}$. This means that every nonempty subset of $\mathbb{Z}^n_{\geq 0}$ has a smallest element under $>$. In other words, if $A \subseteq \mathbb{Z}^n_{\geq 0}$ is nonempty, then there is $\alpha \in A$ such that $\beta > \alpha$ for every $\beta \neq \alpha$ in A.

Given a monomial ordering $>$, we say that $\alpha \geq \beta$ when either $\alpha > \beta$ or $\alpha = \beta$. The following lemma will help us understand what the well-ordering condition of part (iii) of the definition means.

Lemma 1.16. *An order relation $>$ on $\mathbb{Z}^n_{\geq 0}$ is a well-ordering if and only if every strictly decreasing sequence in $\mathbb{Z}^n_{\geq 0}$.*

$$\alpha(1) > \alpha(2) > \alpha(3) > \cdots$$

eventually terminates.

Our first example of an ordering on n-tuples will be lexicographic order (or *lex* order, for short).

Definition 1.17. *(Lexicographic order).* Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ be in $\mathbb{Z}^n_{\geq 0}$. We say $\alpha >_{lex} \beta$ if the leftmost nonzero entry of the vector difference $\alpha - \beta \in \mathbb{Z}^n$ is positive. We will write $x^\alpha >_{lex} x^\beta$ if $\alpha >_{lex} \beta$.

$$(1, 0, \ldots, 1) >_{lex} (0, 1, 0, \ldots, 0) >_{lex} \cdots >_{lex} (0, \ldots, 0, 1).$$

so $x_1 >_{lex} x_2 >_{lex} \cdots >_{lex} x_n$.

The lexicographic order satisfies the three conditions of Definition 1.15.

Proposition 1.18. *The lex ordering in $\mathbb{Z}^n_{\geq 0}$ is a monomial ordering.*

Next, we define the graded lexicographic order (or *grlex* order).

Definition 1.19. *(Graded Lex Order).* Let $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$. We say $\alpha >_{grlex} \beta$ if

$$|\alpha| = \sum_{i=1}^{n} \alpha_i > |\beta| = \sum_{i=1}^{n} \beta_i, \text{ or } |\alpha| = |\beta| \text{ and } \alpha >_{lex} \beta.$$

The grlex ordering satisfies three conditions of Definition 1.15.

Another order on monomials is the graded reverse lexicographical order (or *grevlex* order).

Definition 1.20. *(Graded Reverse Lex Order).* Let $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$. We say $\alpha >_{grevlex} \beta$ if

$$|\alpha| = \sum_{i=1}^{n} \alpha_i > |\beta| = \sum_{i=1}^{n} \beta_i, \text{ or } |\alpha| = |\beta| \text{ and the rightmost nonzero entry of } \alpha - \beta \in \mathbb{Z}^n \text{ is negative}$$

The grevlex ordering satisfies three conditions of Definition 1.15.

1.5 Monomial ideals and Dickson's lemma

To start, we define monomial ideal in $K[x_1, \ldots, x_n]$.

Definition 1.21. An ideal $I \subseteq K[x_1, \ldots, x_n]$ is a *monomial ideal* if there is a subset $A \subseteq \mathbb{Z}^n_{\geq 0}$ (possibly infinite) such that I consists of all polynomials which are finite sums of the form $\sum_{\alpha \in A} h_\alpha x^\alpha$, where $h_\alpha \in K[x_1, \ldots, x_n]$. In this case, we write $I = \langle x^\alpha : \alpha \in A \rangle$.

We first need to characterize all monomials that lie in a given monomial ideal.

Lemma 1.22. *Let $I = \langle x^{\alpha} : \alpha \in A \rangle$ be a monomial ideal. Then a monomial ideal x^{β} lies in I if and only if x^{β} is divisible by x^{α} for some $\alpha \in A$.*

Let us next show that whether a given polynomial f lies in a monomial ideal can be determined by looking at the monomials of f.

Lemma 1.23. *Let I be a monomial ideal, and let $f \in K[x_1, \ldots, x_n]$. Then the following are equivalent:*

(i) $f \in I$.

(ii) Every term of f lies in I.

(iii) f is a k-linear combination of the monomials in I.

We have the following corollary from above lemma.

Corollary 1.24. *Two monomial ideals are the same if and only if they contain the same monomials.*

The main result of this section is that all monomial ideals of $K[x_1, \ldots, x_n]$ are finitely generated.

Proposition 1.25. (Dickson's Lemma) *Let $I = \langle x^{\alpha} : \alpha \in A \rangle \subseteq K[x_1, \ldots, x_n]$ be a monomials ideal. Then I can can be written in the form $I = \langle x^{\alpha(1)}, \ldots, x^{\alpha(s)} \rangle$, where $\alpha(1), \ldots, \alpha(s) \in A$. In particular, I has a finite basis.*

By using Dickson's Lemma, we obtain the following important fact about monomial ordering in $K[x_1, \ldots, x_n]$.

Corollary 1.26. *Let $>$ be a relation on $\mathbb{Z}^n_{\geq 0}$ satisfying:*

(i) $>$ is a total ordering on $\mathbb{Z}^n_{\geq 0}$.

(ii) If $\alpha > \beta$ and $\gamma \in \mathbb{Z}^n_{\geq 0}$, then $\alpha + \gamma > \beta + \gamma$.

Then $>$ is well-ordering if and only if $\alpha \geq 0$ for all $\alpha \in \mathbb{Z}^n_{\geq 0}$.

1.6 The Hilbert basis theorem and Gröbner basis

In this section, we define the Gröbner basis. At first, we define ideal of leading terms, namely *initial ideal*, as follows.

Definition 1.27. Let $I \subseteq K[x_1, \ldots, x_n]$ be an ideal other than $\{0\}$, and fix a monomial ordering on $K[x_1, \ldots, x_n]$. Then:

(i) We denote by $\mathrm{LT}(I)$ the set of leading terms of nonzero elements of I. Thus,

$$\mathrm{LT}(I) = \{cx^{\alpha} : \text{there exists } f \in I \setminus \{0\} \text{ with } \mathrm{LT}(f) = cx^{\alpha}\}.$$

(ii) We denote by $\langle \mathrm{LT}(I) \rangle$ the ideal generated by the elements of $\mathrm{LT}(I)$.

Since $\langle \mathrm{LT}(I) \rangle$ is a monomial ideal, we can apply the result of previous section. In particular, $\langle \mathrm{LT}(I) \rangle$ is generated by finitely many leading terms.

Proposition 1.28. *Let $I \subseteq K[x_1, \ldots, x_n]$ be an ideal different from $\{0\}$.*

(i) $\langle \mathrm{LT}(I) \rangle$ is a monomial ideal.

(ii) There are $g_1, \ldots, g_t \in I$ such that $\langle \mathrm{LT}(g_1), \ldots, \mathrm{LT}(g_t) \rangle$.

By using Proposition 1.28 and the division algorithm, we have following proposition.

Proposition 1.29. ***(Hilbert Basis Theorem).*** *Every ideal $I \subseteq K[x_1, \ldots, x_n]$ has a finite generating set. In other words, $I = \langle g_1, \ldots, g_t \rangle$ for some $g_1, \ldots, g_t \in I$.*

We are now in a position to define Gröbner basis.

Definition 1.30. Fix a monomial order on the polynomial ring $K[x_1, \ldots, x_n]$. A finite subset $G = \{g_1, \ldots, g_t\}$ of an ideal $I \subseteq K[x_1, \ldots, x_n]$ different from $\{0\}$ is said to be a *Gröbner basis* (or *standard basis*) if

$$\langle \mathrm{LT}(g_1), \ldots, \mathrm{LT}(g_t) \rangle = \langle \mathrm{LT}(I) \rangle.$$

Using the convention that $\langle \emptyset \rangle = \{0\}$, we define the empty set $\emptyset$ to be the Gröbner basis of the zero ideal $\{0\}$.

Equivalently, but more informally, a set $\{g_1, \ldots, g_t\} \subseteq I$ is a Gröbner basis of I if and only if the leading term of any element of I is divisible by one of the $\mathrm{LT}(g_i)$.

Corollary 1.31. *Fix a monomial order. Then every ideal $I \subseteq K[x_1, \ldots, x_n]$ has a Gröbner basis. Furthermore, any Gröbner basis for an ideal I is a basis of I.*

Chapter 2

Graph theory

In this chapter, we study about graph theory based on [7].

2.1 Basic terms of graphs

A graph is pair $G = (V, E)$ of sets such that $E \subseteq [V]^2$; thus, the elements of E are 2-element subsets of V. To avoid notational ambiguities, we shall always assume tacitly that $V \cup E = \emptyset$. The elements of V are the *vertices* of the graph G, the elements of E are its *edges*. The usual way to picture a graph is by drawing a dot for each vertex and joining two of these dots by a line if the corresponding two vertices from an edge. Just how these dots and lines are drawn is considered irrelevant: all that matters is the information of which pairs of vertices from an edge and which do not.

A graph with vertex set V is said to be a graph *on* V. The vertex set of a graph G is referred to as $V(G)$, its edge set as $E(G)$.

Two or more edges connecting same two vertices are called *multiple edges*. An edge connecting one vertex is *loop*. *Simple graph* is a graph which has no loops or multiple edges. Throughout this thesis, all graphs are assumed to be finite and simple.

Two vertices x, y of G are *adjacent*, or *neighbours*, if $\{x, y\}$ is an edge of G. Two edges $e \neq f$ are adjacent if they have an end in common. The *degree* $\deg(v)$ of a vertex v is the number of neighbours of v. A vertex of degree 0 is *isolated*.

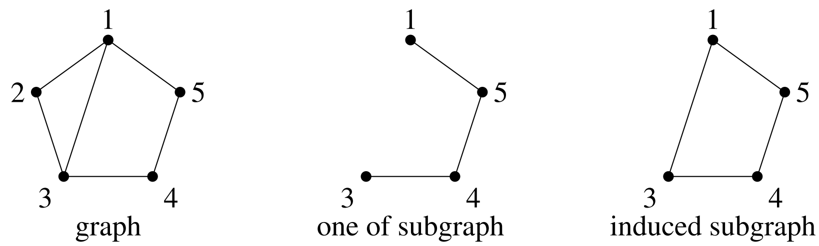
We set $G \cup G' := (V \cup V', E \cup E')$ and $G \cap G' := (V \cap V', E \cap E')$. If $V' \subseteq V$ and $E' \subseteq E$, then G' is a subgraph of G.

Definition 2.1. For $S \subset V$, the *induced subgraph* $G[S]$ of G is a subgraph of G on the vertex set S whose edge set is $\{\{i, j\} \in E : i, j \in S\}$.

Example 2.2. Let G be a graph on $V = \{1, \ldots, 5\}$ and edge set E is

$$\{\{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{1,5\}, \{1,3\}\}.$$

We can obtain many graphs as a subgraph of G. Suppose that $S = \{1, 3, 4, 5\}$. Then, an induced subgraph $G[S]$ is uniquely decided.



A vertex v of a connected graph G is called a *cut vertex* if the graph obtained by the removal of v from G is disconnected. Given a graph G, a *block* of G is a maximal connected subgraph of G without cut vertices.

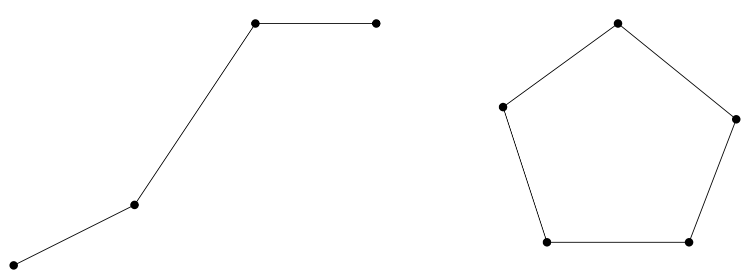
2.2 Classes of graphs

A path is a non-empty graph $P = (V,E)$ of the form

$$V = \{x_0, x_1, \dots, x_k\} \qquad E = \{\{x_0, x_1\}, \{x_1, x_2\}, \dots, \{x_{k-1}, x_k\}\},$$

where the x_i are all distinct. The vertices x_0 and x_k are *linked* by P and are called its *ends*; the vertices $x_0, x_1 \dots, x_k$ are the *inner* vertices of P. The number of edges of a path is its *length*, and the path of length k is denoted by P_k. A graph G is called *connected* if it any two of its vertices are linked by a path in G.

For $n \geq 3$, if $V(P) = \{x_0, x_1, \dots, x_{n-1}\}$ is a vertex set of path P and x_0 and x_{n-1} are ends of P, then the graph $C := P + \{x_0, x_{n-1}\}$ is called a *cycle*. As with paths, the *length* of a cycle of its edges; the cycle of length n is called a n-cycle and denoted by C_n. An edge which joins two vertices of a cycle but is not itself an edge of the cycle of that cycle is a *chord* of its cycle.

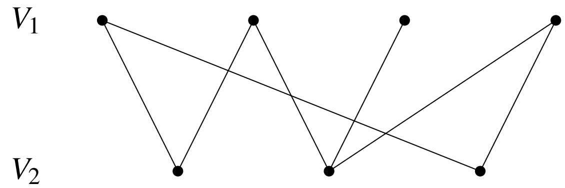


P_3 $\qquad$ C_5

A connected graph which has at most one cycle is called a *pseudotree*.

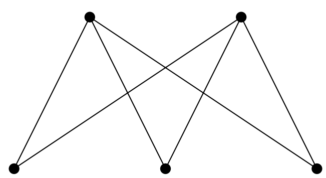
A graph G is called *k-partite* if the vertex set of G can be partitioned into k different independent sets $V_1, \dots, V_k$. When $k = 2$, it is called a *bipartite graph*. Clearly, a bipartite graph cannot contain an *odd cycle*, a cycle of odd length. In fact, the bipartite graphs are characterized by this property.

Proposition 2.3. *A graph G is bipartite if and only if it contains no odd cycle, a cycle of odd length.*

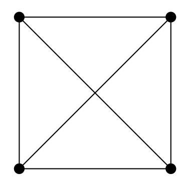


$G = (V_1 \cup V_2, E)$

A *complete k-partite graph* denoted by $K_{|V_1|,\dots,|V_k|}$ is a k-partite graph with a partition $V_1 \cup \cdots \cup V_k$ of its vertex set such that $\{s,t\}$ is an edge of G for any $s \in V_i$, any $t \in V_j$, and any $1 \leq i < j \leq k$. A complete n-partite graph all of whose independent sets V_i have only one vertex is called a *complete graph*, and denoted by K_n. A *clique* is a subset $C \subset V$ such that the induced subgraph $G[C]$ of G is a complete graph.



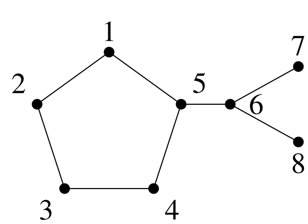
$K_{2,3}$



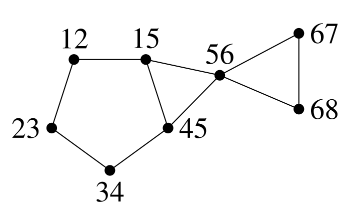
K_4

A vertex coloring of a graph $G = (V,E)$ is a map $c : V \to S$ such that $c(v) \neq c(w)$ whenever v and w are adjacent. The elements of set S are called the available *colours*. The smallest integer k such that G has a k-colouring is the *chromatic number* of G. It is denoted by $\chi(G)$. A graph G with $\chi(G) = k$ is called k-chromatic; if $\chi(G) \leq k$, we call G k-colourable.The greatest integer r such that K_r is a subgraph of G is the *clique number* $\omega(G)$ of G. A graph is called *perfect* if every induced subgraph H of G has chromatic number $\chi(G) = \omega(G)$.

The *line graph* $L(G)$ of $G = (V,E)$ is a graph on the vertex set E with the edge set $\{\{e,e'\} : e,e' \in E, e \cap e' \neq \emptyset\}$.



G

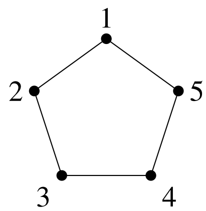


$L(G)$

2.3 Special subsets of vertex set

A finite subset $S \subset V$ is called *stable* in G if none of the edges of G is a subset of S. In particular, the empty set $\emptyset$ is stable.

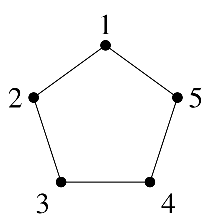
Example 2.4. Let G be a cycle of length 5. The graph G has 11 stable sets.



$\emptyset$,{1},{2},{3},{4},{5}
{1,3},{1,4},{2,4},{2,5},{3,5}

A k-matching of G is a set of k pairwise non-adjacent edges of G. If a matching M includes all vertices of G, then M is called a *perfect matching*. We say that $S \subset V$ induces a *perfectly matchable subgraph* of G if the induced subgraph $G[S]$ of G on the vertex set S has a perfect matching. Let $\mathscr{W}(G)$ be the set of all such subsets of V, and adopt the convention that $\emptyset \in \mathscr{W}(G)$, i.e., that the empty subgraph is perfectly matchable. Given a subset $S \subset V$, let $\rho(S) = \sum_{i \in S} \mathbf{e}_i \in \mathbb{R}^n$, where $\mathbf{e}_i$ is the ith unit vector in $\mathbb{R}^n$. In particular, $\rho(\emptyset)$ is the zero vector.

Example 2.5. Let G be a cycle of length 5. Then the graph G has 11 sets induce perfectly matchable subgraph.



$\emptyset$,{1,2},{1,3},{2,3},{3,4},{4,5},{1,5},
{1,2,3,4},{1,2,4,5},{2,3,4,5},{1,2,3,5}

In this thesis, we study lattice polytope arising from subsets induce perfectly matchable subgraph.

Chapter 3

Convex polytopes

In this chapter, we study convex polytopes based on [14].

3.1 Convex Polytopes and Simplicial Complexes

Let $\mathbb{R}^n$ be a N-dimensional Euclid space

$$\mathbb{R}^n = \{\mathbf{x} = (x_1, \ldots, x_n) : x_i \in \mathbb{R} \text{ for } i = 1, 2, \ldots, n\}.$$

with the standard topology.

A subset $A \subseteq \mathbb{R}^n$ is *convex* if for each $\mathbf{x}, \mathbf{y} \in A$, the line segment joining $\mathbf{x}$ and $\mathbf{y}$ is contained in A.

For every non-empty subset X in $\mathbb{R}^n$ there exists a unique convex set $\mathrm{CONV}(X)$ with $X \subseteq \mathrm{CONV}(X)$ such that if $X \subseteq A \subseteq \mathrm{CONV}(X)$ and A is convex then $A = \mathrm{CONV}(X)$. We say that $\mathrm{CONV}(X)$ is the *convex hull* of X.

Definition 3.1. By a *convex polytope* $\mathscr{P}$ in $\mathbb{R}^n$ we mean the convex hull of a finite set of points in $\mathbb{R}^n$.

The d-ball $\mathbb{B}^d$ in $\mathbb{R}^n$, $d \leq n$ is defined to be

$$\mathbb{B}^d = \left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \begin{array}{l} x_1^2 + x_2^2 + \cdots + x_d^2 \leq 1 \\ x_{d+1} = \cdots = x_N = 0 \end{array} \right\}.$$

Let A and B be subsets in $\mathbb{R}^n$. We write $A \simeq_{homeo} B$ if A is homeomorphic to B, that is, there exists a bijection $\phi : A \to B$ such that both ϕ and ϕ^{-1} are continuous with respect to the standard topology on $\mathbb{R}^n$.

The *dimension*, $\dim \mathscr{P}$, of a convex polytope $\mathscr{P} \subset \mathbb{R}^n$ is the positive integer d with the property $\mathscr{P} \simeq_{homeo} \mathbb{B}^d$.

A *hyperplane* in $\mathbb{R}^n$ is a subset $\mathscr{H} \subset \mathbb{R}^n$ of the form

$$\mathscr{H} = \left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} a_i x_i = b \right\},$$

where each $a_i, b \in \mathbb{R}$ with $(a_1, a_2, \ldots, a_n) \neq (0, 0, \ldots, 0)$. Every hyperplane $\mathscr{H} \subset \mathbb{R}^n$ determines the following two closed half-space in $\mathbb{R}^n$:

$$\mathscr{H}^{(+)} = \left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} a_i x_i \geq b \right\},$$

$$\mathscr{H}^{(-)} = \left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} a_i x_i \leq b \right\},$$

Note that $\mathscr{H}^{(+)} \cap \mathscr{H}^{(-)} = \mathscr{H}$.

Let $\mathscr{P} \subset \mathbb{R}^n$ be a convex polytope of dimension d. A hyperplane $\mathscr{H} \subset \mathbb{R}^n$ is called a *supporting hyperplane* of $\mathscr{P}$ if the following are satisfied:

(i) $\mathscr{P} \subset \mathscr{H}^{(+)}$ or $\mathscr{P} \subset \mathscr{H}^{(-)}$;

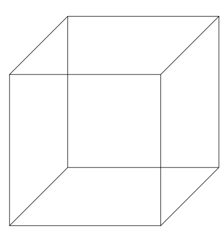
(ii) $\mathscr{P} \cap \mathscr{H} \neq \emptyset$.

Definition 3.2. A *face* of $\mathscr{P}$ is a subset of $\mathscr{P}$ of the form $\mathscr{P} \cap \mathscr{H}$, where $\mathscr{H}$ is a supporting hyperplane of $\mathscr{P}$.

Proposition 3.3. *A convex polytope $\mathscr{P} \subset \mathbb{R}^n$ has only a finite number of faces, and each face is a convex polytope in $\mathbb{R}^n$.*

A face $\mathscr{F}$ of a convex polytope $\mathscr{P} \subset \mathbb{R}^n$ is called an i-face of $\mathscr{P}$ if $\dim \mathscr{F} = i$. Here $\dim \mathscr{F}$ is the dimension of $\mathscr{F}$ as a convex polytope. A point x of $\mathscr{P}$ is called a *vertex* of $\mathscr{P}$ if $\{x\}$ is a 0-face of $\mathscr{P}$ is called an edge of $\mathscr{P}$. Also, when $\dim \mathscr{P} = d$, we say that a $(d-1)$-face of $\mathscr{P}$ is a facet of $\mathscr{P}$.

Example 3.4. A 3-dimensional cube has 6 facets.



Proposition 3.5. *Let V be the set of vertices of $\mathscr{P}$. Then*

(i) $\mathscr{P} = \mathrm{CONV}(V)$*;*

(ii) If $\mathscr{F}$ is a face of $\mathscr{P}$ then $\mathscr{F} = \mathrm{CONV}(V \cap \mathscr{F})$, thus vertex set of $\mathscr{F}$ is equal to $V \cap \mathscr{F}$

The *boundary* of a convex polytope $\mathscr{P} \subset \mathbb{R}^n$ (in the usual topological sense, i.e., in the relative topology on $\mathscr{P}$ from the standard topology on $\mathbb{R}^n$) is denoted by $\partial \mathscr{P}$.

The (d-1)-sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^n$, $d \leq n$, is defined to be

$$\mathbb{S}^{d-1} = \left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \begin{array}{c} x_1^2 + x_2^2 + \cdots + x_d^2 = 1 \\ x_{d+1} = \cdots = x_N = 0 \end{array} \right\}.$$

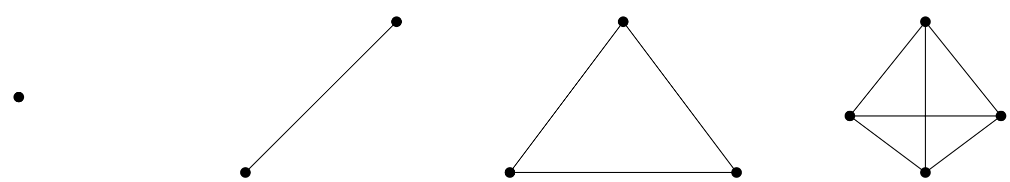
If $\mathscr{P} \subset \mathbb{R}^n$ is a convex polytope of dimension d, then $\delta\mathscr{P} \simeq_{homeo} \mathbb{S}^{d-1}$, since $\mathscr{P} \simeq_{homeo} \mathbb{B}^d$.

Definition 3.6. A *polyhedral complex* in $\mathbb{R}^n$ is a finite set Γ of convex polytopes in $\mathbb{R}^n$ such that

(i) if $\mathscr{P} \in \Gamma$, then each face of $\mathscr{P}$ is in Γ;

(ii) if $\mathscr{P}, \mathscr{Q} \in \Gamma$, then $\mathscr{P} \cap \mathscr{Q}$ is a face of $\mathscr{P}$ and of $\mathscr{Q}$.

A *d-simplex* in $\mathbb{R}^N$ is a convex polytope $\mathscr{P} \subset \mathbb{R}^n$ of dimension d such that the number of $\mathscr{P}$ is just $d+1$.



0-simplex 1-simplex 2-simplex 3-simplex

Definition 3.7. A *simplicial complex* $\triangle$ is a polyhedral complex such that every face σ is a simplex.

Definition 3.8. Let Γ be a polyhedral complex in $\mathbb{R}^n$ of dimension $d-1$. We write $f_i = f_i(\Gamma)$ for the number of i-faces of $\Gamma (i = 0, 1, \ldots, d-1)$. The vector

$$f(\Gamma) := (f_0, f_1, \ldots, f_{d-1}),$$

is called the f-vectors of Γ.

Note that if Γ and Γ' have the same combinatorial type, then we have $f(\Gamma) = f(\Gamma')$. However, the converse is not true.

Definition 3.9. Let Γ be a polyhedral complex of dimension $d-1$ with $f(\Gamma) = (f_0, f_1, \ldots, f_{d-1})$. We define the integers $h_i = h_i(\Gamma)$, $0 \le i \le d$, by the formula

$$\sum_{i=0}^{d} f_{i-1}(x-1)^{d-1} = \sum_{i=0}^{d} h_i x^{d-i}.$$

Here we set $f_{-1} := 1$. We say that the vector $h(\Gamma) := (h_0, h_1, \ldots, h_d)$ is the *h-vector* of Γ.

Note that knowing the f-vector of Γ is equivalent to knowing the h-vector of Γ.

3.2 Unimodular triangulations

Next, We study triangulations of configuration matrix based on [15]. Let $A = (a_{ij})_{1 \le i \le d, 1 \le j \le n}$ be a $d \times n$ matrix and

$$\mathbf{a}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{dj} \end{bmatrix}, 1 \le j \le n$$

the column vectors of A.

Let $\mathbb{Z}$ denote the set of integers and write $\mathbb{Z}^{d \times n}$ for the set of $d \times n$ matrices $A = (a_{ij})_{1 \le i \le d, 1 \le j \le n}$ with each $a_{ij} \in \mathbb{Z}$.

The *inner product* of vectors $\mathbf{a} = [a_1, a_2, \ldots, a_d]^T$ and $\mathbf{b} = [b_1, b_2, \ldots, b_d]^T$, where T stands for the transpose, belonging to $\mathbb{R}^d$ is defined to be

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{d} a_i b_i.$$

A matrix $A = (a_{ij})_{1 \le i \le d, 1 \le j \le n} \in \mathbb{Z}^{d \times n}$ is called a *configuration matrix* if there exists $\mathbf{c} \in \mathbb{R}^n$ such that

$$\mathbf{a}_j \cdot \mathbf{c} = 1, 1 \le j \le n.$$

We regard configuration matrix $A = [\mathbf{a}_1, \ldots, \mathbf{a}_n]$ as a set $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$. Let $\triangle$ be a collection of simplices whose vertices belong to a configuration matrix A. Then, $\triangle$ is called a *covering* of A if holds. In addition, if a covering $\triangle$ of a configuration matrix A is a simplicial complex, then it is called a *triangulation* of A.

For a configuration matrix $A = [\mathbf{a}_1, \ldots, \mathbf{a}_n] \in \mathbb{Z}^{d \times n}$, let

$$\mathbb{Z}A = \left\{ \sum_{i=1}^{n} z_i \mathbf{a}_i : z_i \in \mathbb{Z} \right\} \in \mathbb{Z}^d.$$

Let $B \subset \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ be the vertex set of a maximal simplex $\sigma \in \triangle$ in a covering(triangulation) of A.

Definition 3.10. The *normalized volume* of $\sigma \in \triangle$ is defined by $VOL(\sigma) := [\mathbb{Z}A : \mathbb{Z}B]$, that is, the index of a subgroup $\mathbb{Z}B$ in a group $\mathbb{Z}A$.

A covering(triangulation) $\triangle$ of A is said to be *unimodular* if the normalized volume of any maximal simplex in $\triangle$ is equal to 1.

3.3 Lexicographic triangulations and unimodular configurations

We study configuration matrices which are the base cases for an inductive construction of lexicographic and reverse lexicographic triangulations.

Proposition 3.11. *Let A be a configuration matrix. Suppose that* $\mathrm{CONV}(A)$ *is a simplex and that A is the vertex set of* $\mathrm{CONV}(A)$*. Then, there exists only one triangulation of A, and it is the set of all faces of* $\mathrm{CONV}(A)$*.*

Next, we study lexicographic triangulations. A triangulation $\triangle$ of a configuration matrix $A \in \mathbb{Z}^{d\times n}$ is called a *lexicographic triangulation* if we have $\triangle = \triangle(in_{<_{lex}}(I_A))$ for a lexicographic order $<_{lex}$ induced by an ordering $x_{i_1} > \cdots > x_{i_n}$ of variables. Since toric ideals are homogeneous ideals, there is no difference between lexicographic orders and pure lexicographic orders and pure lexicographic orders for such ideals. It is known that every lexicographic triangulation can be computed recursively, as follows.

Proposition 3.12. *For a configuration matrix $A \in \mathbb{Z}^{d\times n}$, let $\triangle_{lex}(A)$ be a lexicographic triangulation with respect to lexicographic order $<_{lex}$ induced by an ordering $x_1 > \cdots > x_n$ of variables. If $\mathbf{a}_1 \in \mathrm{CONV}(A \setminus \{\mathbf{a}_1\})$, then we have*

$$\triangle_{lex}(A) = \triangle_{lex}(A \setminus \{\mathbf{a}_1\}).$$

In addition, if $a_1 \notin \mathrm{CONV}(A \setminus \{\mathbf{a}_1\})$, then we have

$$\triangle_{lex}(A) = \triangle_{lex}(A \setminus \{\mathbf{a}_1\}) \cup \triangle$$

$$\triangle \left\{ \mathrm{CONV}(\{\mathbf{a}_1\} \cup B) : \begin{array}{c} B \subset A \setminus \{\mathbf{a}_1\}\}) \\ \mathrm{CONV}(B) \in \triangle_{lex}(A \setminus \{\mathbf{a}_1\}) \\ \mathrm{CONV}(B) \textit{is visible from } \mathbf{a}_1 \end{array} \right\}.$$

A configuration matrix A is said to be *unimodular* if all triangulation of A are unimodular. Here "all triangulations" means all regular triangulations and all nonregular triangulations. However, it will be turn out that it is enough to consider only the lexicographic triangulations.

Proposition 3.13. *For a configuration matrix $A \in \mathbb{Z}^{d\times n}$, the following conditions are equivalent.*

(i) A is a unimodular configuration matrix.

(ii) Any triangulation of A is unimodular.

(iii) Any lexicographic triangulation of A is unimodular.

(iv) The normalized volume of any maximal simplex all of whose vertices belong to A is equal to 1*.*

(v) For an arbitrary $f \in \mathscr{C}_A$, any monomial appearing in f is squarefree.

If $\mathrm{rank}(A) = d$*, then the following is also equivalent to the above.*

(vi) All nonzero $d \times d$ minors of A have the same absolute value.

3.4 Reverse lexicographic triangulations and compressed triangulations

A triangulation $\triangle$ of a configuration matrix $A \in \mathbb{Z}^{d\times n}$ is called a *reverse lexicographic triangulation* if $\triangle = \triangle(\text{in}_{rev}(I_A))$ with respect to a reverse lexicographic order $<_{rev}$ induced by the ordering $x_{i_1} > \cdots > x_{i_n}$ of variables. This is sometimes called a "pulling triangulation" in the literature. As in the case of lexicographic triangulations, every reverse lexicographic triangulations can be computed recursively.

A configuration matrix A is said to be *compressed* if, for any ordering $x_{i_1} > \cdots > x_{i_n}$ of n variables of $K[x]$, the reverse lexicographic triangulation $\triangle(\text{in}_{<_{rev}}(I_A))$ of A with respect to the reverse lexicographic order $<_{rev}$ induced by this ordering is unimodular.

3.5 Combinatorics on δ-Sequences

A *lattice polytope* $\mathscr{P} \subset \mathbb{R}^n$ is a convex polytope such that any vertex of $\mathscr{P}$ belongs to $\mathbb{Z}^n$. Suppose that $\mathscr{P} \subset \mathbb{R}^n$ is an lattice polytope of dimension d and let $\partial\mathscr{P}$ be the boundary of $\mathscr{P}$. Given an integer $n > 0$, we set

$$n\mathscr{P} := \{n\alpha \in \mathscr{P}\}$$

and define

$$i(\mathscr{P}, n) := \#(n\mathscr{P} \cap \mathbb{Z}^n),$$
$$i^*(\mathscr{P}, n) := \#[n(\mathscr{P} - \partial\mathscr{P}) \cap \mathbb{Z}^n].$$

Then there exist integers $\delta_0, \delta_1, \ldots, \delta_d$ such that

$$1 + \sum_{n=1}^{\infty} i(\mathscr{P}, n)\lambda^n = \frac{\delta_0 + \delta_1\lambda + \cdots + \delta_d\lambda_d}{(1-\lambda)^{d+1}}.$$

We say that

$$\delta(\mathscr{P}) := (\delta_0, \delta_1, \ldots, \delta_d)$$

is the δ-vectors of $\mathscr{P}$. It is known that:

(i) $\delta_0 = 1, \delta_1 = i(\mathscr{P}, 1) - (d+1) = \#(\mathscr{P} \cap \mathbb{Z}^N) - (d+1)$;

(ii) (Ehrhart's Law pf Reciprocity)

$$\sum_{n=1}^{\infty} i^*(\mathscr{P}, n)\lambda^n = \frac{\delta_d\lambda + \delta_{d-1}\lambda^2 + \cdots + \delta_0\lambda^{d+1}}{(1-\lambda)^{d+1}};$$

(iii) $\delta_d = i^*(\mathscr{P}, 1) = \#[(\mathscr{P} - \partial\mathscr{P}) \cap \mathbb{Z}^N]$;

(iv) (Stanley) $\delta(\mathscr{P}) \geq 0$ i.e., $\delta_i \geq 0$, $\forall i$;

(v) When $N = d$, volume of

$$\mathscr{P} = \frac{\delta_0 + \delta_1 + \cdots + \delta_d}{d!}.$$

An arbitrary (not necessary integral) convex polytope $\mathscr{P} \subset \mathbb{R}^d$ is called of *standard type* if the following conditions are satisfied:

(i) the dimension of $\mathscr{P}$ is equal to d;

(ii) the origin $(0,0,\ldots,0)$ of $\mathbb{R}^n$ is contained in the interior $\mathscr{P} - \partial\mathscr{P}$ of $\mathscr{P}$.

A lattice polytope $\mathscr{P} \subset \mathbb{R}^n$ is said to be *reflexive* if $\mathbf{0}$ is the unique lattice point in its interior and the dual polytope

$$\mathscr{P}^* := \{x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for any } y \in \mathscr{P}\}$$

is again a lattice polytope. Recall that $x \cdot y$ is the inner product of x and y. Note that each vertex of $\mathscr{P}^*$ corresponds to a facet of $\mathscr{P}$.

Proposition 3.14. *Suppose that $\mathscr{P}$ is an lattice polytope of standard type. Then the δ-vector, $\delta = (\delta_0, \delta_1, \ldots, \delta_d)$ of $\mathscr{P}$ is symmetric if and only if the lattice polytope $\mathscr{P}$ is reflexive.*

Chapter 4

Main theorems

4.1 Toric rings and toric ideals

Let $K[\mathbf{x}^{\pm 1}, s] = K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, s]$ be a Laurent polynomial ring in $n+1$ variables over a field K. For a lattice point $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$, we define $\mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in K[\mathbf{x}^{\pm 1}, s]$. If $\mathscr{P} \cap \mathbb{Z}^n = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$, then the *toric ring* $K[\mathscr{P}]$ of $\mathscr{P}$ is the K-subalgebra of $K[\mathbf{x}^{\pm 1}, s]$ generated by the monomials $\mathbf{x}^{\mathbf{a}_1} s, \ldots, \mathbf{x}^{\mathbf{a}_m} s \in K[\mathbf{x}^{\pm 1}, s]$. Furthermore, the *toric ideal* $I_{\mathscr{P}}$ is the defining ideal of $K[\mathscr{P}]$, i.e., the kernel of a surjective ring homomorphism $\pi : K[y_1, \ldots, y_m] \to K[\mathscr{P}]$ defined by $\pi(y_i) = \mathbf{x}^{\mathbf{a}_i} s$ for $i = 1, 2, \ldots, m$. It is known that $I_{\mathscr{P}}$ is generated by homogeneous binomials. See, e.g., [13, 38] for details.

A lattice polytope $\mathscr{P}$ is called compressed if the initial ideal of $I_{\mathscr{P}}$ is generated by squarefree monomials with respect to any reverse lexicographic order [39]. It is known that [38, Corollary 8.9] the initial ideal of $I_{\mathscr{P}}$ is generated by squarefree monomials if and only if the corresponding triangulation of $\mathscr{P}$ using only the lattice points in $\mathscr{P}$ is unimodular. Hence $\mathscr{P}$ is compressed if and only if every pulling triangulation of $\mathscr{P}$ using only the lattice points in $\mathscr{P}$ is unimodular. Sullivant [39] proved that a lattice polytope is compressed if and only if it is 2-*level*, which is important in optimization theory. For example, the convex polytope of all $n \times n$ doubly stochastic matrices, hypersimplices, the order polytopes of finite posets, edge polytopes of bipartite graphs and complete multipartite graphs, and the stable set polytopes of perfect graphs are compressed.

On the other hand, $\mathscr{P} \subset \mathbb{R}^n$ is said to be *normal* if $K[\mathscr{P}]$ is a normal semigroup ring. It is known that

- $\mathscr{P}$ is normal if and only if every vector in $k\mathscr{P} \cap L_{\mathscr{P}}$ is a sum of k vectors from $\mathscr{P} \cap \mathbb{Z}^n$, where $L_{\mathscr{P}}$ is the sublattice of $\mathbb{Z}^n$ spanned by $\mathscr{P} \cap \mathbb{Z}^n$;
- $\mathscr{P}$ is normal if there exists a monomial order such that the initial ideal of $I_{\mathscr{P}}$ is generated by squarefree monomials. In particular, $\mathscr{P}$ is normal if $\mathscr{P}$ is compressed.

A lattice polytope $\mathscr{P} \subset \mathbb{R}^n$ has the *integer decomposition property* (IDP) if every vector in $k\mathscr{P} \cap \mathbb{Z}^n$ is a sum of k vectors from $\mathscr{P} \cap \mathbb{Z}^n$. In particular, $\mathscr{P}$ is normal if $\mathscr{P}$ has IDP. However, the converse does not hold in general.

Two lattice polytopes $\mathscr{P} \subset \mathbb{R}^n$ and $\mathscr{P}' \subset \mathbb{R}^{n'}$ are said to be *unimodularly equivalent* if there exists an affine map from the affine span

$$\mathrm{aff}(\mathscr{P}) = \left\{ \sum_{i=1}^{r} \lambda_i \alpha_i : 1 \leq r \in \mathbb{Z}, \alpha_i \in \mathscr{P}, \lambda_i \in \mathbb{R}, \sum_{i=1}^{r} \lambda_i = 1 \right\}$$

of $\mathscr{P}$ to the affine span $\mathrm{aff}(\mathscr{P}')$ of $\mathscr{P}'$ that maps $\mathbb{Z}^n \cap \mathrm{aff}(\mathscr{P})$ bijectively onto $\mathbb{Z}^{n'} \cap \mathrm{aff}(\mathscr{P}')$ and that maps $\mathscr{P}$ to $\mathscr{P}'$. A lattice polytope $\mathscr{P} \subset \mathbb{R}^n$ of dimension n is called *Gorenstein of index* δ if $\delta\mathscr{P} = \{\delta a : a \in \mathscr{P}\}$ is unimodularly equivalent to a reflexive polytope. In particular, a reflexive polytope is Gorenstein of index 1. Reflexive polytopes are related to mirror symmetry and studied in many areas of mathematics. They are key combinatorial tools for constructing topologically mirror-symmetric pairs of Calabi-Yau varieties, as shown by Batyrev [3]. It is known that a lattice polytope $\mathscr{P}$ is Gorenstein if and only if the Ehrhart ring

$$K[\mathbf{x}^{\alpha} s^m : \alpha \in mP \cap \mathbb{Z}^n, m \in \mathbb{Z}_{\geq 0}] \subset K[\mathbf{x}^{\pm 1}, s]$$

of $\mathscr{P}$ is Gorenstein. On the other hand, the Ehrhart ring of $\mathscr{P}$ coincides with the toric ring of $\mathscr{P}$ if and only if $\mathscr{P}$ has IDP.

4.2 Perfectly matchable subgraph polytopes

In the present thesis, we study conditions for perfectly matchable subgraph polytopes to be compressed or Gorenstein. Let $G = (V, E)$ be a graph on the vertex set $V = [n] := \{1, 2, \ldots, n\}$ and the edge set E.

Recall that $S \subset V$ induces a perfectly matchable subgraph of G if the induced subgraph $G[S]$ of G on the vertex set S has a perfect matching and $\mathscr{W}(G)$ be the set of all such subsets of V.The *perfectly matchable subgraph polytope* of G, denoted by $\mathscr{P}_G$, is the convex hull of $\{\rho(S) \in \mathbb{R}^n : S \in \mathscr{W}(G)\}$.

The perfectly matchable subgraph polytope of a graph is defined in [1]. The motivation of their study on perfectly matchable subgraph polytopes is to solve optimization problems that arise in practice (e.g., a bus driver scheduling problem). In optimization theory, compressed polytopes are important since semidefinite programming relaxations are very efficient for compressed polytopes (see, e.g, [11]).

Recently, perfectly matchable subgraphs of graphs appear in the study of h^*-polynomials of lattice polytopes.

If G is a bipartite graph with a partition $V_1 \cup V_2 = [n]$, let $\widehat{G}$ be a connected bipartite graph on $[n+2]$ whose edge set is $E(\widehat{G}) = E(G) \cup \{\{i, n+1\} : i \in V_1\} \cup \{\{j, n+2\} : j \in V_2 \cup \{n+1\}\}$. It is known [33, Proposition 3.4] that

$$I_{\widehat{G}}(x) = \sum_{S \in \mathscr{W}(G)} x^{|S|},$$

where $I_{\widehat{G}}(x)$ is the interior polynomial of $\widehat{G}$ that is introduced by Kálmán [17] as a version of the Tutte polynomials for hypergraphs. It was shown [18] that the h^*-polynomial of the edge polytope of a bipartite graph G coincides with the interior polynomial $I_G(x)$ of a hypergraph induced by G. Using these facts, several results on h^*-polynomials of several important classes of lattice polytopes are obtained [6, 33, 34, 35].

Next, we introduce inequalities for the facets of $\mathscr{P}_G$ given in [1, 2]. We will see that these inequalities depend on whether G is bipartite. Let

$$\mathscr{T} = \{X \subset V : \text{each component of } G[X] \text{ has an odd number of vertices}\}.$$

For any $A \subset V$, let $\Gamma(A)$ denote the subset of $V \setminus A$ that consists of vertices adjacent to at least one vertex in A. For any $S \subset V$, let $\theta(S)$ be the number of connected components of the induced subgraph $G[S]$.

Proposition 4.1 ([2]). *Let $G = (V,E)$ be a graph. Then $\mathscr{P}_G$ is a set of vectors $x \in \mathbb{R}^V$ such that*

$$0 \le x(v) \le 1 \text{ for all } v \in V \tag{4.1}$$
$$x(S) - x(\Gamma(S)) \le |S| - \theta(S) \tag{4.2}$$

for all $S \in \mathscr{T}$ such that every component of $G[S]$ consists of a single vertex or else is a nonbipartite graph with an odd number of vertices.

A graph $G = (V,E)$ is called *critical* (or *factor-critical*) if, for every $v \in V$, $G \setminus \{v\}$ has a perfect matching. Any critical graph is either a single vertex or a connected nonbipartite graph with an odd number of vertices.

Proposition 4.2 ([2]). *Let G be a nonbipartite graph. For $S \in \mathscr{T}$, the inequality (4.2) is facet-inducing for $\mathscr{P}_G$ if and only if S satisfies the following conditions*:

(i) *every component of $G[S]$ is critical*;

(ii) *every component of $G \setminus (S \cup \Gamma(S))$ is nonbipartite*;

(iii) *the graph obtained from $G[S \cup \Gamma(S)]$ by deleting all edges with both ends in $\Gamma(S)$ is connected.*

Remark that, if $|S| = 1$, then S satisfies conditions (i) and (iii).

Proposition 4.3 ([1]). *Let $G = (V,E)$ be a bipartite graph on the vertex set $V = V_1 \cup V_2$. Then $\mathscr{P}_G$ is a set of vectors $x \in \mathbb{R}^V$ such that*

$$0 \le x(v) \le 1 \text{ for all } v \in V, \tag{4.3}$$
$$x(S) - x(\Gamma(S)) \le 0 \text{ for all } \emptyset \ne S \subset V_1, \tag{4.4}$$
$$x(V_1) - x(V_2) = 0. \tag{4.5}$$

Proposition 4.4 ([1]). *Let $G = (V_1 \cup V_2, E)$ be a connected bipartite graph. Then the following are true:*

(a) *The inequality* $0 \le x(v)$ *is facet-inducing for* $\mathscr{P}_G$ *if and only if* v *is not a cut vertex. (Note that every vertex of degree one is not a cut vertex.)*

(b) *Suppose that* G *has at least two edges. Then the inequality* $x(v) \le 1$ *is facet-inducing for* $\mathscr{P}_G$ *if and only if* $\deg(v) \ge 2$;

(c) *For any* $\emptyset \ne S \subsetneq V_1$*, the inequality (4.4) is facet-inducing for* $\mathscr{P}_G$ *if and only if both* $G[S \cup \Gamma(S)]$ *and* $G[(V_1 \setminus S) \cup (V_2 \setminus \Gamma(S))]$ *are connected.*

Remark 4.5. Suppose that both $G[S \cup \Gamma(S)]$ and $G[(V_1 \setminus S) \cup (V_2 \setminus \Gamma(S))]$ are connected for $S \subsetneq V_1$. Let $S' = V_2 \setminus \Gamma(S)$. Then we have $\Gamma(S') = V_1 \setminus S$. If (4.5) holds, then we have $x(S) - x(\Gamma(S)) = x(S') - x(\Gamma(S'))$.

4.3 Main theorems

If G is the disjoint union of graphs G_1 and G_2, then we have

$$\mathscr{P}_G = \mathscr{P}_{G_1} \times \mathscr{P}_{G_2} := \{(x,y) : x \in \mathscr{P}_{G_1}, y \in \mathscr{P}_{G_2}\}$$

since every matching M of G is of the form $M = M_1 \cup M_2$ where M_i is a matching of G_i for each $i = 1,2$. Hence, $\mathscr{P}_G$ is compressed if and only if both $\mathscr{P}_{G_1}$ and $\mathscr{P}_{G_2}$ are compressed. On the other hand, from [10, Corollary 4.3.3 and Theorem 4.4.9] (cf. [19, Lemma 2.9]), when both $\mathscr{P}_{G_1}$ and $\mathscr{P}_{G_2}$ have IDP, $\mathscr{P}_G$ is Gorenstein of index δ if and only if both $\mathscr{P}_{G_1}$ and $\mathscr{P}_{G_2}$ are Gorenstein of index δ. Thus, when we are studying such properties, we may assume that G is connected.

The first main result of the present thesis is a complete characterization of compressed perfectly matchable subgraph polytopes.

Theorem 4.6. *Let* G *be a connected graph. Then* $\mathscr{P}_G$ *is compressed if and only if all blocks of* G *are complete bipartite graphs except for at most one block, which is either* K_4 *or* $K_{1,1,n}$.

In particular, if $\mathscr{P}_G$ is compressed, then the line graph of G is perfect by Proposition 6.3.

The second main result of the present thesis is a characterization of Gorenstein perfectly matchable subgraph polytopes of bipartite graphs. For any $S \subset V$, let $\Gamma(S)$ denote the subset of $V \setminus S$ that consists of vertices adjacent to at least one vertex in S. A graph is called *k-connected* if any induced subgraph obtained by removing less than k vertices is connected. Theorem 4.7 follows from Proposition 5.3, Theorem 7.4 and Corollary 7.6.

Theorem 4.7. *Suppose that a connected bipartite graph* $G = (V_1 \cup V_2, E)$ *has a vertex* v *with* $\deg(v) \ge 2$ *such that* v *is not a cut vertex. Then the following conditions are equivalent*:

(i) $K[\mathscr{P}_G]$ *is Gorenstein*;

(ii) *$\mathscr{P}_G$ is Gorenstein*;

(iii) *$\mathscr{P}_G$ is Gorenstein of index* 2;

(iv) *G has a perfect matching and, for any subset $\emptyset \neq S \subsetneq V_1$ such that $G[S \cup \Gamma(S)]$ and $G[(V_1 \setminus S) \cup (V_2 \setminus \Gamma(S))]$ are connected, we have $|S| + 1 = |\Gamma(S)|$.*

Moreover, if G is 2-connected, then the above conditions are equivalent to

(v) *the edge polytope of G is Gorenstein.*

The third main result of the present thesis is a complete characterization of Gorenstein perfectly matchable subgraph polytopes of pseudotrees.

A *bidegreed graph* is a graph with two different vertex degrees. For example, a path P_n and a star graph $K_{1,n-1}$ are bidegreed if $n \geq 3$.

Theorem 4.8. *Let G be a pseudotree on the vertex set V. Then $K[\mathscr{P}_G]$ is Gorenstein if and only if G satisfies one of the following*:

(i) *G is K_1, K_2, or a bidegreed tree*;

(ii) $G = C_5$;

(iii) *G has a triangle C and*

$$\begin{cases} \deg(v) \in \{2,3\} & \textit{if } v \in V(C) \\ \deg(v) \in \{1,3\} & \textit{otherwise}; \end{cases}$$

(iv) *G has an even cycle C, and there exists an integer $\delta \geq 2$ such that*

$$\begin{cases} \deg(v) = \delta & \textit{if } v \in V(C) \\ \deg(v) \in \{1, \delta - 1\} & \textit{otherwise.} \end{cases}$$

The relationships between toric rings of perfectly matchable subgraph polytopes and toric rings of other polytopes play important roles in this thesis. In [1], it was pointed out that $\mathscr{P}_G$ is unimodulary equivalent to a base polytope of a transversal matroid if G is bipartite. Let G be a graph with the edge set $E = \{e_1, \dots, e_m\}$, and let

$$A_G = (\rho(e_1), \dots, \rho(e_m))$$

be the matrix associated with the edge polytope $\mathrm{Ed}(G)$ of G. It then follows that

$$\mathscr{P}_G \cap \mathbb{Z}^n = \{A_G\, x : x \in \mathrm{Stab}(L(G)) \cap \mathbb{Z}^m\}, \tag{4.6}$$

where $\mathrm{Stab}(L(G))$ is the stable set polytope of the line graph of G (definitions are explained later). From (4.6), it is easy to see that the edge polytope of G is normal if $\mathscr{P}_G$ is normal (Corollary 5.6). There are many research on the edge polytopes and the stable set polytopes from the point of view of not only discrete geometry but also combinatorial commutative algebra. The study of perfectly matchable subgraph polytopes is expected to contribute the study of these polytopes.

Chapter 5

Relationships with toric rings of other polytopes

In this chapter, we introduce relationships between toric rings of $\mathscr{P}_G$ and toric rings of other polytopes, i.e., base polytopes of matroids, stable set polytopes, and edge polytopes.

Let $G = (V,E)$ be a graph. Recall that the perfectly matchable subgraph polytope $\mathscr{P}_G$ of G is the convex hull of $\{\rho(S) \in \mathbb{R}^n : S \in \mathscr{W}(G)\}$, where $\mathscr{W}(G)$ is the set of all subsets of V which induce perfectly matchable subsets of G.

Example 5.1. Let G be a cycle C_4 of length 4. Then the set $\mathscr{W}(G)$ associated with G is

$$\mathscr{W}(G) = \{\emptyset, \{1,2\}, \{2,3\}, \{3,4\}, \{1,4\}, \{1,2,3,4\}\}.$$

Note that matchings $\{\{1,2\},\{3,4\}\}$ and $\{\{1,4\},\{2,3\}\}$ are associated with the same set $\{1,2,3,4\}$. The perfectly matchable subgraph polytope $\mathscr{P}_G \subset \mathbb{R}^4$ of G is the convex hull of the column vectors of the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

Then $\mathscr{P}_G$ is a 3-dimensional polytope which is compressed and Gorenstein.

It is known [1, 2] that, if G is a connected graph on the vertex set $\{1,2,\ldots,n\}$, then

$$\dim \mathscr{P}_G = \begin{cases} n-1 & \text{if } G \text{ is bipartite,} \\ n & \text{otherwise.} \end{cases}$$

If $G = (V,E)$ is a bipartite graph on the vertex set $V = \{1,2,\ldots,n\} = V_1 \cup V_2$, then $\mathscr{P}_G$ is contained in the hyperplane

$$\{x \in \mathbb{R}^n : x(V_1) = x(V_2)\},$$

where $x(V_i)$ denotes the sum $\sum_{j \in V_i} x_j$ for the vector $x = (x_1,\ldots,x_n)$. If G is the disjoint union of graphs G_1 and G_2, then $\mathscr{P}_G$ is the product of $\mathscr{P}_{G_1}$ and $\mathscr{P}_{G_2}$ and hence $\dim(\mathscr{P}_G) = \dim(\mathscr{P}_{G_1}) + \dim(\mathscr{P}_{G_2})$. If G has k connected components which are bipartite, then we have $\dim \mathscr{P}_G = n-k$.

5.1 Base polytopes of transversal matroids

Suppose that, for $\mathscr{B} = \{B_1, \dots, B_n\}$, B_i is a subset of $[d]$ and each $|B_i|$ has r elements for $i = 1, \dots, n$. We define $\mathscr{I} = \{B' \subset [d] : B' \subset B_i \text{ for some } i\}$

Definition 5.2. If a set $\mathscr{B}$ satisfies condition:

For each $1 \leq i, j \leq n$, there exists $y \in B_j \setminus B_i$ satisfies $(B_i \setminus \{x\}) \cup \{y\} \in \mathscr{B}$ for every $x \in B_i \setminus B_j$

we call $M = ([d], \mathscr{I})$ matroid of rank r on a ground set $[d]$

Let M be a matroid on a ground set $\{1, 2, \dots, n\}$ with the set of bases $\mathscr{B}$. The *base polytope* $B(M)$ of M is the convex hull of the set $\{\rho(B) \ : \ B \in \mathscr{B}\} \subset \mathbb{R}^n$. In [1], it was pointed out that $\mathscr{P}_G$ is unimodularly equivalent to a base polytope of a transversal matroid if G is bipartite. Since the base polytope of any matroid has IDP [42], we have the following.

Proposition 5.3. *Let G be a connected bipartite graph. Then $\mathscr{P}_G$ has IDP. In particular, $K[\mathscr{P}_G]$ is Gorenstein if and only if $\mathscr{P}_G$ is Gorenstein.*

Note that Gorenstein base polytopes of matroids were studied in [19].

5.2 Edge polytopes

Let G be a graph on the vertex set $V = \{1, 2, \dots, n\}$ and the edge set $E = \{e_1, \dots, e_m\}$. The *edge polytope* $\mathrm{Ed}(G)$ of G is the convex hull of the set

$$\{\rho(e_1), \dots, \rho(e_m)\}.$$

Note that compressed (resp. Gorenstein) edge polytopes were studied in [27, 30] (resp. [32]). Toric rings of edge polytopes, namely *edge rings*, are studied intensively. See, e.g., [13, 41] for details. In particular, the edge ring of a finite connected simple graph with a q-linear resolution, where $q \geq 3$, is a hypersurface [16, 26]. We say that a graph G satisfies the *odd cycle condition* if, for any two odd cycles C_1 and C_2 in the same connected component of G without common vertices, there exists an edge $\{i, j\}$ of G such that $i \in V(C_1)$ and $j \in V(C_2)$.

Proposition 5.4 ([29, 36])**.** *Let G be a graph. Then* $\mathrm{Ed}(G)$ *is normal if and only if G satisfies the odd cycle condition.*

5.3 Stable set polytopes

Recall that a finite subset $S \subset V$ is called stable in G if none of the edges of G is a subset of S. Let $S(G) = \{S_1, \dots, S_t\}$ denote the set of all stable sets of G. The *stable set polytope* of G is the convex hull of the set $\{\rho(S_1), \dots, \rho(S_t)\}$, denoted by $\mathrm{Stab}(G)$.

It is known (e.g., [31]) that $\mathrm{Stab}(G)$ is compressed if and only if G is perfect. Moreover, it is known [32, Theorem 1.2 (b)] that, for any perfect graph G, $\mathrm{Stab}(G)$ is Gorenstein if and only if all maximal cliques of G have the same cardinality.

For $S \in \mathscr{W}(G)$, we have $\rho(S) = \rho(e_{i_1}) + \cdots + \rho(e_{i_k}) \in \mathbb{R}^n$, where $\{e_{i_1}, \dots, e_{i_k}\} \subset E$ is a matching of G. Note that $\{e_{i_1}, \dots, e_{i_k}\} \subset E$ is a matching of G if and only if $S' = \{e_{i_1}, \dots, e_{i_k}\} \subset V(L(G))$ is stable in $L(G)$. Since $\rho(S') = \mathbf{e}_{i_1} + \cdots + \mathbf{e}_{i_k} \in \mathbb{R}^m$, we have

$$\rho(S) = A_G\, \rho(S'),$$

where $A_G = (\rho(e_1), \dots, \rho(e_m))$ is the vertex-edge incidence matrix of G. Thus

$$\mathscr{P}_G \cap \mathbb{Z}^n = \{A_G\, x : x \in \mathrm{Stab}(L(G)) \cap \mathbb{Z}^m\}. \tag{5.1}$$

In addition, we have $\mathrm{Ed}(G) \subset \mathscr{P}_G$. In such a case, the following holds in general.

Proposition 5.5. *Let $\mathscr{P} \subset \mathscr{P}' \subset \mathbb{R}^n$ be lattice polytopes such that*

$$\begin{aligned} \mathscr{P} \cap \mathbb{Z}^n &= \{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m\}, \\ \mathscr{P}' \cap \mathbb{Z}^n &= \{(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m)\, x : x \in Q\}, \end{aligned}$$

where $Q \subset \mathbb{Z}^m_{\geq 0}$. Suppose that there exists $\mathbf{w} \in \mathbb{R}^n$ such that $\mathbf{a}_i \cdot \mathbf{w} = 1$ for any $1 \leq i \leq m$. Then $\mathscr{P}$ is normal if $\mathscr{P}'$ is normal.

Proof. Let $\alpha \in k\mathscr{P} \cap L_{\mathscr{P}}$. Since $\mathscr{P} \subset \mathscr{P}'$, α belongs to $k\mathscr{P}' \cap L_{\mathscr{P}'}$. The normality of $\mathscr{P}'$ guarantees that $\alpha = \alpha_1 + \cdots + \alpha_k$ where $\alpha_i \in \mathscr{P}' \cap \mathbb{Z}^n$ for each i. Then $\alpha_i = (\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m)x$ for some nonnegative integer vector $x \in Q$. Hence each α_i is a sum of vectors from $\mathscr{P} \cap \mathbb{Z}^n$ if x is not zero. Thus we have $\alpha = \mathbf{a}_{j_1} + \cdots + \mathbf{a}_{j_\ell}$ for some $1 \leq j_1, \dots, j_\ell \leq m$. Since there exists $\mathbf{w} \in \mathbb{R}^n$ such that $\mathbf{a}_i \cdot \mathbf{w} = 1$ for any $1 \leq i \leq m$, it follows that $k = \mathbf{w} \cdot \alpha = \mathbf{w} \cdot \mathbf{a}_{j_1} + \cdots + \mathbf{w} \cdot \mathbf{a}_{j_\ell} = \ell$. Thus α is a sum of k vectors from $\mathscr{P} \cap \mathbb{Z}^n$, as desired. □

Since $\rho(e_i) \cdot \mathbf{w} = 1$ $(1 \leq i \leq m)$ for $\mathbf{w} = (1/2, \dots, 1/2)$, we have the following.

Corollary 5.6. *Let G be a graph. If $\mathscr{P}_G$ is normal, then $\mathrm{Ed}(G)$ is normal (i.e., G satisfies the odd cycle condition).*

Given a lattice polytope $\mathscr{P} \subset \mathbb{R}^n$, let $A = (\mathbf{a}_1, \dots, \mathbf{a}_m)$ where $\mathscr{P} \cap \mathbb{Z}^n = \{\mathbf{a}_1, \dots, \mathbf{a}_m\}$. It is known [38] that the toric ideal $I_{\mathscr{P}}$ of $\mathscr{P}$ is generated by binomials $\mathbf{y}^{\mathbf{a}} - \mathbf{y}^{\mathbf{b}} \in K[y_1, \dots, y_m]$ such that $A(\mathbf{a} - \mathbf{b}) = \mathbf{0}$ and $\deg \mathbf{y}^{\mathbf{a}} = \deg \mathbf{y}^{\mathbf{b}}$.

Proposition 5.7. *Let G be a pseudotree which has no even cycles. Then the toric ring $K[\mathscr{P}_G]$ of $\mathscr{P}_G$ is isomorphic to the toric ring $K[\mathrm{Stab}(L(G))]$ of $\mathrm{Stab}(L(G))$.*

Proof. Let $G = (V,E)$ be a graph on the vertex set $V = \{1,2,\ldots,n\}$ and the edge set $E = \{e_1,\ldots,e_m\}$. Let $A_G = (\rho(e_1),\ldots,\rho(e_m))$ be the vertex-edge incidence matrix of G and let $B_G = (\rho(S_1),\ldots,\rho(S_r))$ where $\{S_1,\ldots,S_r\}$ is the set of all stable sets of $L(G)$.

If G is not a tree, then $n = m$ and hence A_G is an $m \times m$ matrix. It is known [13, Lemmas 5.5 and 5.6] that A_G is a regular matrix if and only if G is a pseudotree which has no even cycles. If G is a tree, then $n = m+1$ and A_G is an $(m+1) \times m$ matrix of rank m.

Since the rank of the $n \times m$ matrix A_G is m, for any $\mathbf{u} \in \mathbb{R}^r$, $A_G B_G \mathbf{u} = \mathbf{0}$ if and only if $B_G \mathbf{u} = \mathbf{0}$. From (5.1), $\mathscr{P}_G \cap \mathbb{Z}^n = \{A_G\rho(S_1),\ldots,A_G\rho(S_r)\}$. Hence we have

$$\begin{aligned}
\mathbf{y}^{\mathbf{a}} - \mathbf{y}^{\mathbf{b}} \in I_{\mathscr{P}_G} &\Leftrightarrow A_G B_G(\mathbf{a}-\mathbf{b}) = \mathbf{0} \text{ and } \deg \mathbf{y}^{\mathbf{a}} = \deg \mathbf{y}^{\mathbf{b}} \\
&\Leftrightarrow B_G(\mathbf{a}-\mathbf{b}) = \mathbf{0} \text{ and } \deg \mathbf{y}^{\mathbf{a}} = \deg \mathbf{y}^{\mathbf{b}} \\
&\Leftrightarrow \mathbf{y}^{\mathbf{a}} - \mathbf{y}^{\mathbf{b}} \in I_{\mathrm{Stab}(L(G))}.
\end{aligned}$$

Thus $K[\mathscr{P}_G] \simeq K[\mathbf{y}]/I_{\mathscr{P}_G} = K[\mathbf{y}]/I_{\mathrm{Stab(L(G))}} \simeq K[\mathrm{Stab}(L(G))]$ as desired. □

Recall that a *clique* C of a graph G is a subset of the vertex set of G such that the induced subgraph $G[C]$ of G is a complete graph. The graph obtained by gluing two graphs at a clique of them is called a *clique-sum* of them. An *almost bipartite graph* is a graph whose induced subgraph obtained by deleting a vertex v is bipartite for some v.

Proposition 5.8. *Let G be a pseudotree. Then $\mathscr{P}_G$ is normal.*

Proof. From Proposition 5.3, we may assume that G is not bipartite. Then, by Proposition 5.7, $K[\mathscr{P}_G] \cong K[\mathrm{Stab}(L(G))]$. Hence it is enough to show that $\mathrm{Stab}(L(G))$ is normal. We will show it by induction on the number of vertices of G. Let n (≥ 3) be the number of vertices of G.

If $n = 3$, then $G = K_3$ since G has an odd cycle. Thus $L(G) = K_3$, and hence $\mathrm{Stab}(L(G))$ is a simplex. Then $\mathrm{Stab}(L(G))$ is normal.

Suppose that $n > 3$ and that, for any pseudotree G' with $\leq n-1$ vertices, $\mathrm{Stab}(L(G'))$ is normal. If G is an odd cycle, then $L(G)$ is also an odd cycle. It is known [8, Theorem 8.1] that the toric ideal of the stable set polytope of an almost bipartite graph has a squarefree quadratic initial ideal. Hence, $\mathrm{Stab}(L(G))$ is normal. Suppose that G is not an odd cycle. Then G has a vertex v such that $\deg(v) = 1$. Let e be the edge $\{v,v'\}$ of G. Let E be the edge set of G and let $E' = \{e' \in E : v' \in e'\}$. Then $L(G)$ is the clique-sum of the graphs G_1 and G_2, where G_1 is the induced subgraph of $L(G)$ obtained by the vertex set E', and G_2 is the induced subgraph of $L(G)$ obtained by the vertex set $E \setminus \{e\}$. In fact, $E' \setminus \{e\} = E' \cap (E \setminus \{e\})$ is a clique of both G_1 and G_2. Since G_1 is a complete graph, $\mathrm{Stab}(G_1)$ is a simplex, and hence normal. Since G_2 is the line graph of pseudotree $G \setminus \{v\}$ with $n-1$ vertices, $\mathrm{Stab}(G_2)$ is normal by the hypothesis of induction. It is known [23, Proposition 1] that the stable set polytope of the clique-sum of simple graphs G_1 and G_2 is normal if and only if both $\mathrm{Stab}(G_1)$ and $\mathrm{Stab}(G_2)$ are normal. It then follows that $\mathrm{Stab}(L(G))$ is normal. □

Chapter 6

Compressed perfectly matchable subgraph polytopes

In this chapter, we prove Theorem 4.6. The following proposition is due to Sullivant [39, Theorem 2.4] (and also appeared in Haase's dissertation [12]).

Proposition 6.1 ([39])**.** *Let $\mathscr{P}$ be a lattice polytope having the irredundant linear description $\mathscr{P} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}_i \cdot \mathbf{x} \geq b_i,\ i = 1, \ldots, s\}$, where $\mathbf{a}_i \in \mathbb{R}^n$ for $i = 1, 2, \ldots, s$. In addition, let $\mathscr{L} \subset \mathbb{Z}^n$ be a lattice spanned by $\mathscr{P} \cap \mathbb{Z}^n$. Then $\mathscr{P}$ is compressed if and only if, for each i, there is at most one nonzero $m_i \in \mathbb{R}$ such that $\{\mathbf{x} \in \mathscr{L} : \mathbf{a}_i \cdot \mathbf{x} = b_i + m_i\} \cap \mathscr{P} \neq \emptyset$.*

Let G' be an induced subgraph of a graph G. Then $\mathscr{P}_{G'}$ is a face of $\mathscr{P}_G$. It is known that every face of a compressed polytope is compressed. Hence, we have the following immediately.

Lemma 6.2. *Let G' be a connected graph such that $\mathscr{P}_{G'}$ is not compressed. If a graph G has G' as an induced subgraph, then $\mathscr{P}_G$ is not compressed.*

The following fact is known in graph theory.

Proposition 6.3 ([22, 40])**.** *Let G be a graph. Then the following conditions are equivalent*:

(i) *The line graph $L(G)$ of G is perfect*;

(ii) *G has no odd cycle of length ≥ 5 as a subgraph*;

(iii) *Each block of G is either a bipartite graph, K_4, or $K_{1,1,n}$.*

It is known that $\mathrm{Stab}(G)$ is compressed if and only if G is perfect.

Lemma 6.4. *Let G be a connected graph. If $\mathscr{P}_G$ is compressed, then $L(G)$ is perfect and hence* $\mathrm{Stab}(L(G))$ *is compressed.*

Proof. Suppose that $L(G)$ is not perfect. From Proposition 6.3, G has an odd cycle C_{2n+1} with $n \geq 2$ as a subgraph. Let $S = V(C_{2n+1})$ and $G' = G[S]$. From Lemma 6.2, it is enough to prove that $\mathscr{P}_{G'}$ arising from the induced subgraph G' of G is not compressed. By assumption, $G' \setminus (S \cup \Gamma(S))$ is empty. Since C_{2n+1} is critical and since the vertex set of C_{2n+1} coincides with the vertex set of G', the graph $G' = G'[S]$ is critical. Moreover, $\Gamma(S) = \emptyset$ and $G'[S \cup \Gamma(S)] = G'$ is connected. Hence,

$$x(S) \leq |S| - \theta(S) = 2n+1-1 = 2n$$

is facet-inducing for $\mathscr{P}_{G'}$ from Proposition 4.2. Then there exist $n+1 \geq 3$ kinds of values for $x(S)$ with $x \in \mathscr{P}_{G'} \cap \mathbb{Z}^{2n+1}$. In fact, we have

$$\left\{x(S) : x \in \mathscr{P}_{G'} \cap \mathbb{Z}^{2n+1}\right\} = \{0, 2, \dots, 2n\}.$$

From Proposition 6.1, $\mathscr{P}_{G'}$ is not compressed. □

Lemma 6.5. *Let G be a connected graph. Suppose that $\mathscr{P}_G$ is compressed. Then for any even cycle C in G of length $2n \geq 6$, the induced subgraph $G[V(C)]$ is a complete bipartite graph $K_{n,n}$.*

Proof. Suppose that $\mathscr{P}_G$ is compressed. Let $C = (v_1, \dots, v_{2n})$ be an even cycle in G of length $2n \geq 6$, and let $G' = G[V(C)]$. We prove the statement by induction on n.

From Lemma 6.2, $\mathscr{P}_{G'}$ is compressed. Note that G' is a subgraph of a block of G. From Proposition 6.3 and Lemma 6.4, G' is a bipartite graph since neither K_4 nor $K_{1,1,n}$ has an even cycle of length ≥ 6. Let $V_1' = \{v_1, v_3, \dots, v_{2n-1}\}$ and $V_2' = \{v_2, v_4, \dots, v_{2n}\}$. Suppose that G' is not a complete bipartite graph.

Case 1. ($n = 3$) Suppose that $\{v_3, v_6\}$ is not an edge of G'. Then, for $S = \{v_3\}$, both $G'[S \cup \Gamma(S)] = G'[\{v_2, v_3, v_4\}]$ and $G'[(V_1' \setminus S) \cup (V_2' \setminus \Gamma(S)) = G'[\{v_1, v_5, v_6\}]$ are connected. However, we have

$$x(S) - x(\Gamma(S)) = \begin{cases} 0 & \text{for } x = \rho(\emptyset) = \mathbf{0}, \\ -1 & \text{for } x = \rho(\{v_1, v_2\}), \\ -2 & \text{for } x = \rho(\{v_1, v_2, v_4, v_5\}). \end{cases}$$

Hence, $\mathscr{P}_{G'}$ is not compressed, a contradiction. It follows that C has all the possible chords $\{v_1, v_4\}$, $\{v_2, v_5\}$, and $\{v_3, v_6\}$. Hence, G' is a complete bipartite graph $K_{3,3}$.

Case 2. ($n \geq 4$ and suppose that the statement is true for any even cycle of length $\leq 2n-2$)

Suppose that $\{v_3, v_{2k}\}$ for some $3 \leq k \leq n$ is not an edge of G'. If $\{v_3, v_{2k'}\}$ is an edge of G' for some k', then v_3, v_{2k} are contained in an even cycle of length $2m$ with $6 \leq 2m \leq 2n-2$. By the hypothesis of induction, $\{v_3, v_{2k}\}$ is an edge of G', a contradiction. Thus, for any $3 \leq k' \leq n$, $\{v_3, v_{2k'}\}$ is not an edge of G'. Let $S = \{v_3\}$. Then both $G'[S \cup \Gamma(S)] = G'[\{v_2, v_3, v_4\}]$ and $G'[(V_1' \setminus S) \cup (V_2' \setminus \Gamma(S))] = G'[\{v_1, v_5, v_6, \dots, v_{2n}\}]$ are connected. However, we have

$$x(S) - x(\Gamma(S)) = \begin{cases} 0 & \text{for } x = \rho(\emptyset) = \mathbf{0}, \\ -1 & \text{for } x = \rho(\{v_1, v_2\}), \\ -2 & \text{for } x = \rho(\{v_1, v_2, v_4, v_5\}). \end{cases}$$

Hence, $\mathscr{P}_{G'}$ is not compressed, a contradiction. Thus, G' is a complete bipartite graph $K_{n,n}$. □

Lemma 6.6. *Let G be a connected graph. If $\mathscr{P}_G$ is compressed, then any two triangles of G have a common edge.*

Proof. Suppose that $\mathscr{P}_G$ is compressed and two triangles C and C' of G have no common edges.

Case 1. (C and C' have exactly one common vertex) Let $G' = C \cup C'$, where $C = (v_1, v_2, v_3)$ and $C' = (v_3, v_4, v_5)$. Then the vertex set and the edge set of G' are

$$V' = \{v_1, v_2, v_3, v_4, v_5\},\ E' = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_1\}, \{v_3, v_4\}, \{v_4, v_5\}, \{v_5, v_3\}\}.$$

Since G has no odd cycle of length ≥ 5 as a subgraph, G' is an induced subgraph of G, and hence $\mathscr{P}_{G'}$ is compressed.

We now consider the facets of $\mathscr{P}_{G'}$. Let $S = \{v_3\}$. Since $|S| = 1$, the set S satisfies conditions (i) and (iii) in Lemma 4.2. In addition, since $G' \setminus (S \cup \Gamma(S))$ is empty, S satisfies condition (ii) in Lemma 4.2. Thus, S induces a facet of $\mathscr{P}_{G'}$. However, we have

$$x(S) - x(\Gamma(S)) = \begin{cases} 0 & \text{for } x = \rho(\emptyset) = \mathbf{0}, \\ -2 & \text{for } x = \rho(\{v_1, v_2\}), \\ -4 & \text{for } x = \rho(\{v_1, v_2, v_4, v_5\}). \end{cases}$$

Hence, $\mathscr{P}_{G'}$ is not compressed, a contradiction.

Case 2. (C and C' have no common vertices) From Case 1, we may assume that there is no pair of triangles having exactly one common vertex. Since G is connected, there exists a path $P = (v_1, p_1, \ldots, p_s = v'_1)$ connecting two triangles $C = (v_1, v_2, v_3)$ and $C' = (v'_1, v'_2, v'_3)$, where $p_1, \ldots, p_{s-1} \notin V(C) \cup V(C')$. We may assume that $s \geq 1$ is minimal among pairs of triangles without common edges. Let G'' be an induced subgraph on the vertex set $V(C) \cup V(C') \cup V(P)$. Let $S = \{v_2\}$. Since $|S| = 1$, the set S satisfies conditions (i) and (iii) in Lemma 4.2.

Case 2.1. ($\Gamma(S) = \{v_1, v_3\}$) For this case, $G'' \setminus (S \cup \Gamma(S))$ is nonbipartite. However, we have

$$x(S) - x(\Gamma(S)) = \begin{cases} 0 & \text{for } x = \rho(\emptyset) = \mathbf{0}, \\ -1 & \text{for } x = \rho(\{v_1, p_1\}), \\ -2 & \text{for } x = \rho(\{v_1, v_3\}). \end{cases}$$

Hence, $\mathscr{P}_{G''}$ is not compressed, a contradiction.

Case 2.2. ($\Gamma(S) \neq \{v_1, v_3\}$) There exists an edge $\{v_2, v\}$, where v ($\neq v_1$) belongs to either the path P or C'. If $v \neq p_1$, then an odd cycle of length ≥ 5 is a subgraph of G''. Hence, we have $v = p_1$. Then G'' has triangles (v_1, v_2, v) and (v'_1, v'_2, v'_3) connected by a path $(v = p_1, \ldots, p_s = v'_1)$ ($s \geq 2$). This contradicts the hypothesis that s is minimal. □

We are now in a position to prove a main theorem.

Proof of Theorem 4.6. ("Only if") Suppose that $\mathscr{P}_G$ is compressed. From Proposition 6.3 and Lemma 6.4, each block of G is either a bipartite graph, K_4, or $K_{1,1,n}$. By Lemma 6.6, at most one block is either K_4 or $K_{1,1,n}$. It is enough to show that each bipartite block is a complete bipartite graph. Let B be a bipartite block of G on the vertex set $B_1 \cup B_2$. Suppose that $\{i,j\}$ is not an edge of G for vertices $i \in B_1$ and $j \in B_2$. Since B is 2-connected, there exist two disjoint paths P_1 and P_2 from i to j in B. Note that the length of each P_i is at least 3. Hence $P_1 \cup P_2$ is an even cycle of length ≥ 6. This contradicts to Lemma 6.5. Thus, B is a complete bipartite graph.

("if") Suppose that all blocks of G are complete bipartite graphs except for at most one block, which is either K_4 or $K_{1,1,n}$ and $\mathscr{P}_G$ is not compressed.

Case 1. (G is bipartite) There exists $\emptyset \neq S \subsetneq V_1$ such that

$$x(S) - x(\Gamma(S)) \leq 0$$

is facet-inducing, and

$$x(S) - x(\Gamma(S)) \leq -2$$

for some $x \in \mathscr{P}_G \cap \mathbb{Z}^n$. It then follows that there exist four distinct vertices $i, i' \in \Gamma(S)$ and $j, j' \in V_1 - S$ such that $\{i,j\}, \{i',j'\} \in E$. By Proposition 4.4, $G[S \cup \Gamma(S)]$ and $G[(V_1 - S) \cup (V_2 - \Gamma(S))]$ are connected. Then there exists an even cycle

$$C = (i, j, k_1, \ldots, k_{2p-1}, j', i', \ell_1, \ldots, \ell_{2q-1}),$$

where

$$k_1, k_3, \ldots, k_{2p-1} \in V_2 - \Gamma(S), \quad k_2, k_4, \ldots, k_{2p-2} \in V_1 - S,$$
$$\ell_1, \ell_3, \ldots, \ell_{2q-1} \in S, \quad \ell_2, \ell_4, \ldots, \ell_{2q-2} \in \Gamma(S).$$

Note that the length of C is at least 6. However, since $G[V(C)]$ is complete bipartite, G has an edge $\{k_1, \ell_1\}$. This contradicts $k_1 \in V_2 - \Gamma(S)$ and $\ell_1 \in S$.

Case 2. (G is not bipartite) There exists a subset $S \subset V$ such that

$$x(S) - x(\Gamma(S)) \leq |S| - \theta(S) \tag{6.1}$$

is facet-inducing, and

$$|\{x(S) - x(\Gamma(S)) \in \mathbb{Z} : x \in \mathscr{P}_G \cap \mathbb{Z}^n\}| \geq 3.$$

Note that G has no odd cycle of length ≥ 5 as a subgraph.

Case 2.1. (S is stable and $G = G[S \cup \Gamma(S)]$) Since $G = G[S \cup \Gamma(S)]$, $x(S) - x(\Gamma(S))$ must be even. It then follows that there exists $x \in \mathscr{P}_G \cap \mathbb{Z}^n$ such that

$$x(S) - x(\Gamma(S)) \leq |S| - \theta(S) - 4 = -4.$$

Hence, $G[\Gamma(S)]$ has two edges $\{i_1, j_1\}$ and $\{i_2, j_2\}$ without a common vertex. Since the graph G' obtained from G by deleting all edges with both ends in $\Gamma(S)$ is connected, there exists a path P in G' from i_1 to j_1. By assumption, S is stable, and hence G' is a bipartite graph. Since i_1 and j_1 belong to the same part in the bipartite graph G', the length of P is even. Then $T_1 = P \cup \{i_1, j_1\}$ is an odd cycle of G. Since G has no odd cycle of length ≥ 5, $T_1 = (i_1, j_1, k_1)$ for some $k_1 \in S$. By the same argument, G has a triangle $T_2 = (i_2, j_2, k_2)$ for some $k_2 \in S$. Hence G has two triangles T_1 and T_2 without a common edge. This is a contradiction.

Case 2.2. (S is stable and $G \neq G[S \cup \Gamma(S)]$) From $|S| = \theta(S)$, $x(S) - x(\Gamma(S)) \leq 0$ is facet-inducing. By Proposition 4.2, every component of $G \setminus (S \cup \Gamma(S))$ is nonbipartite, and hence has a triangle. Since any two triangles of G have a common edge, it follows that $G \setminus (S \cup \Gamma(S))$ is connected. If $G[\Gamma(S)]$ has an edge $\{i, j\}$, $G[S \cup \Gamma(S)]$ has a triangle as in Case 2.1. This is a contradiction. Hence $\Gamma(S)$ is a stable set. Thus $G[S \cup \Gamma(S)]$ is a connected bipartite graph. Since $x(S) - x(\Gamma(S)) \leq -2$ for some $x \in \mathscr{P}_G \cap \mathbb{Z}^n$, it follows that there exist four distinct vertices $i, i' \in \Gamma(S)$ and $j, j' \in V \setminus (S \cup \Gamma(S))$ such that $\{i, j\}, \{i', j'\} \in E$. Since $G[S \cup \Gamma(S)]$ and $G \setminus (S \cup \Gamma(S))$ are connected, there exists a cycle

$$C = (i, j, k_1, \ldots, k_p, j', i', \ell_1, \ldots, \ell_{2q-1}),$$

where $p \geq 0$, $q \geq 1$ and

$$k_1, k_2, \ldots, k_p \in V \setminus (S \cup \Gamma(S)),$$
$$\ell_1, \ell_3, \ldots, \ell_{2q-1} \in S, \quad \ell_2, \ell_4, \ldots, \ell_{2q-2} \in \Gamma(S).$$

Note that the length of C is at least 5. Since G has no odd cycle of length ≥ 5 as a subgraph, C is an even cycle of length ≥ 6 (and hence $p \geq 1$ is odd). However, since $G[V(C)]$ is complete bipartite, G has an edge $\{k_1, \ell_1\}$. This contradicts $k_1 \in V \setminus (S \cup \Gamma(S))$ and $\ell_1 \in S$.

Case 2.3. (S is not stable) Since every component of $G[S]$ is critical, a component of $G[S]$ has a triangle T_1 and other components of $G[S]$ are isolated vertices. Let G' be the component of $G[S]$ that contains T_1. It is known [21] that a graph is critical if and only if each block of the graph is critical. Hence every block of G' is critical. Note that (i) any critical graph is either a single vertex or a nonbipartite graph with an odd number of vertices; (ii) $K_{1,1,s}$ is critical if and only if $s = 1$. Since G' is an induced subgraph of G, it then follows that $G' = K_3$ $(= T_1)$. Hence $|S| - \theta(S) = 2$.

By Lemma 6.6, $G \setminus (S \cup \Gamma(S))$ has no triangle. Thus, either $G \setminus (S \cup \Gamma(S))$ is bipartite or $G \setminus (S \cup \Gamma(S)) = \emptyset$. Since (6.1) is facet-inducing, $G \setminus (S \cup \Gamma(S)) = \emptyset$, i.e., $G = G[S \cup \Gamma(S)]$.

If $G[\Gamma(S)]$ has an edge $\{i, j\}$, then G has a triangle $T_2 = (i, j, k)$ for some $k \in S$ as in Case 2.1. Hence G has two triangles T_1 and T_2 without a common edge. This is a contradiction. Thus $\Gamma(S)$ is stable.

Since $G = G[S \cup \Gamma(S)]$, $x(S) - x(\Gamma(S))$ must be even. It then follows that there exists $x \in \mathscr{P}_G \cap \mathbb{Z}^n$ such that

$$x(S) - x(\Gamma(S)) \leq |S| - \theta(S) - 4 = -2.$$

However, since $\Gamma(S)$ is stable, $x(S) - x(\Gamma(S)) \geq 0$ which is a contradiction. $\square$

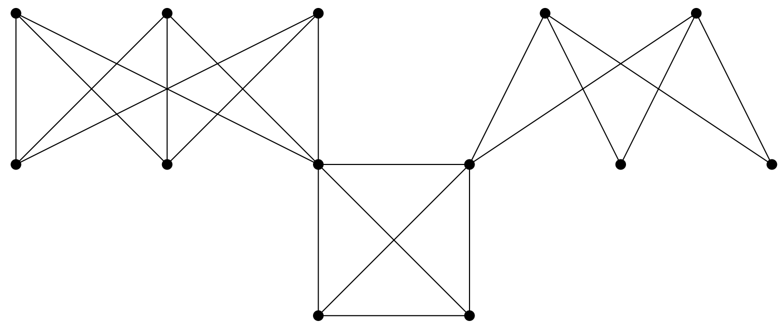


Figure 6.1: Graph which has $K_{3,3}$, K_4, and $K_{2,3}$ as blocks

Example 6.7. The perfectly matchable subgraph polytope $\mathscr{P}_G$ of the graph G in Figure 6.1 is compressed.

Chapter 7

Gorenstein perfectly matchable subgraph polytopes

In this chapter, for several classes of graphs, we give a characterization of a graph G such that $K[\mathscr{P}_G]$ is Gorenstein. If G is either K_1 or K_2, then $K[\mathscr{P}_G]$ is isomorphic to a polynomial ring and hence $K[\mathscr{P}_G]$ is Gorenstein. Throughout this chapter, we may assume that G has at least two edges.

7.1 2-connected bipartite graphs

Suppose that G is a bipartite graph on the vertex set $V = V_1 \cup V_2 = \{1, \ldots, n\}$, where $n \in V_2$. Then $\mathscr{P}_G$ lies on the hyperplane $\mathscr{H}$ defined by the equation $x(V_1) = x(V_2)$. Let $\psi : \mathbb{R}^{n-1} \to \mathscr{H}$ denote the affine map defined by setting

$$\psi(y) = (y_1, \ldots, y_{n-1}, y(V_1) - y(V_2 \setminus \{n\})),$$

for each $y = (y_1, \ldots, y_{n-1}) \in \mathbb{R}^{n-1}$. Then ψ is an affine isomorphism such that $\psi(\mathbb{Z}^{n-1}) = \mathscr{H} \cap \mathbb{Z}^n$. Hence, $\psi^{-1}(\mathscr{P}_G) \subset \mathbb{R}^{n-1}$ is a lattice polytope of dimension $n-1$ which is unimodulary equivalent to $\mathscr{P}_G$.

If G is bipartite, we have the following criterion for G whose $\mathscr{P}_G$ is Gorenstein. Note that any vertex of degree one is not a cut vertex.

Proposition 7.1. *Let G be a connected bipartite graph on the vertex set $V = \{1, 2, \ldots, n\} = V_1 \cup V_2$. Then $\mathscr{P}_G$ is Gorenstein of index δ if and only if $\delta \geq 2$ and there exists $\alpha \in \mathbb{Z}^n$ such that the following hold:*

(i) $\alpha(V_1) = \alpha(V_2)$;

(ii) If v is not a cut vertex, then $\alpha(v) = 1$;

(iii) If $\deg(v) \geq 2$, then $\alpha(v) = \delta - 1$;

(iv) If $G[S \cup \Gamma(S)]$ and $G[(V_1 \setminus S) \cup (V_2 \setminus \Gamma(S))]$ are connected for a subset $\emptyset \neq S \subsetneq V_1$, then $\alpha(S) - \alpha(\Gamma(S)) = -1$.

Proof. Let $\mathscr{P} = \psi^{-1}(\mathscr{P}_G)$, where ψ is the map defined as above. Then $\mathscr{P}$ is Gorenstein of index δ if and only if there exists a lattice point $\beta \in \delta(\mathscr{P} \setminus \partial\mathscr{P}) \cap \mathbb{Z}^{n-1}$ such that $\delta\mathscr{P} - \beta$ is a reflexive polytope, where $\delta\mathscr{P} = \{\delta a : a \in \mathscr{P}\}$ and $\partial\mathscr{P}$ is the boundary of $\mathscr{P}$.

By Propositions 4.3, 4.4 and Remark 4.5, substituting $x(n) = x(V_1) - x(V_2 \setminus \{n\})$, it follows that $\delta\mathscr{P}$ is a set of vectors $x \in \mathbb{R}^{V\setminus\{n\}}$ such that

$$0 \le x(v) \le \delta \qquad \text{for all } v \in V \setminus \{n\}, \tag{7.1}$$

$$0 \le x(V_1) - x(V_2 \setminus \{n\}) \le \delta, \tag{7.2}$$

$$x(S) - x(\Gamma(S)) \le 0 \qquad \begin{array}{l}\text{for all } \emptyset \ne S \subsetneq V_1 \text{ such that } n \notin \Gamma(S) \text{ and}\\ G[S \cup \Gamma(S)] \text{ and } G[(V_1 \setminus S) \cup (V_2 \setminus \Gamma(S))]\\ \text{are connected,}\end{array} \tag{7.3}$$

$$-x(V_1 \setminus S) + x(V_2 \setminus \Gamma(S)) \le 0 \qquad \begin{array}{l}\text{for all } \emptyset \ne S \subsetneq V_1 \text{ such that } n \in \Gamma(S) \text{ and}\\ G[S \cup \Gamma(S)] \text{ and } G[(V_1 \setminus S) \cup (V_2 \setminus \Gamma(S))]\\ \text{are connected.}\end{array} \tag{7.4}$$

Furthermore, $\delta\mathscr{P} - \beta$, where $\beta \in \delta(\mathscr{P} \setminus \partial\mathscr{P}) \cap \mathbb{Z}^{n-1}$, is a set of vectors $x \in \mathbb{R}^{V\setminus\{n\}}$ such that

$$-\beta(v) \le x(v) \le \delta - \beta(v) \text{ for all } v \in V \setminus \{n\},$$

$$-\beta(V_1) + \beta(V_2 \setminus \{n\}) \le x(V_1) - x(V_2 \setminus \{n\}) \le \delta - \beta(V_1) + \beta(V_2 \setminus \{n\}),$$

$$x(S) - x(\Gamma(S)) \le -\beta(S) + \beta(\Gamma(S)) \qquad \begin{array}{l}\text{for all } \emptyset \ne S \subsetneq V_1 \text{ such that } n \notin \Gamma(S) \text{ and}\\ G[S \cup \Gamma(S)] \text{ and } G[(V_1 \setminus S) \cup (V_2 \setminus \Gamma(S))]\\ \text{are connected,}\end{array}$$

$$-x(V_1 \setminus S) + x(V_2 \setminus \Gamma(S)) \le \beta(V_1 \setminus S) - \beta(V_2 \setminus \Gamma(S)) \qquad \begin{array}{l}\text{for all } \emptyset \ne S \subsetneq V_1 \text{ such that } n \in \Gamma(S) \text{ and}\\ G[S \cup \Gamma(S)] \text{ and } G[(V_1 \setminus S) \cup (V_2 \setminus \Gamma(S))]\\ \text{are connected.}\end{array}$$

Note that $0 < \beta(v) < \delta$ and $-\beta(S) + \beta(\Gamma(S) \setminus \{n\}) > 0$ for all $\emptyset \ne S \subsetneq V_1$. Thus, $\mathscr{P}$ is Gorenstein if and only if $\delta \ge 2$ and there exists $\beta \in \delta(\mathscr{P} \setminus \partial\mathscr{P}) \cap \mathbb{Z}^{n-1}$ such that

$$\begin{array}{rcl}
-\beta(v) & = -1 & \text{if } v\ (\ne n) \text{ is not a cut vertex}\\
\delta - \beta(v) & = 1 & \text{if } \deg(v) \ge 2, v \ne n\\
-\beta(V_1) + \beta(V_2 \setminus \{n\}) & = -1 & \text{if } n \text{ is not a cut vertex}\\
\delta - \beta(V_1) + \beta(V_2 \setminus \{n\}) & = 1 & \text{if } \deg(n) \ge 2\\
-\beta(S) + \beta(\Gamma(S)) & = 1 & \text{if } n \notin \Gamma(S) \text{ and } G[S \cup \Gamma(S)] \text{ and } G[(V_1 \setminus S) \cup (V_2 \setminus \Gamma(S))]\\
 & & \text{are connected for a subset } \emptyset \ne S \subsetneq V_1,\\
\beta(V_1 \setminus S) - \beta(V_2 \setminus \Gamma(S)) & = 1 & \text{if } n \in \Gamma(S) \text{ and } G[S \cup \Gamma(S)] \text{ and } G[(V_1 \setminus S) \cup (V_2 \setminus \Gamma(S))]\\
 & & \text{are connected for a subset } \emptyset \ne S \subsetneq V_1.
\end{array}$$

By taking $\alpha = \psi(\beta)$, this is equivalent to conditions (i)–(iv). □

The following characterization is known for graphs having a perfect matching.

Proposition 7.2 (Hall's marriage theorem)**.** *Let G be a bipartite graph on the vertex set $V_1 \cup V_2$. Then G has a perfect matching if and only if $|V_1| = |V_2|$ and $|S| \le |\Gamma(S)|$ for any subset $\emptyset \neq S \subset V_1$.*

Recall that a lattice polytope $\mathscr{P} \subset \mathbb{R}^n$ is said to be *Gorenstein of index* 2 if there exists a lattice point $\alpha \in 2(\mathscr{P} \setminus \partial \mathscr{P}) \cap \mathbb{Z}^n$ such that $2\mathscr{P} - \alpha$ is a reflexive polytope.

Proposition 7.3. *Let G be a connected bipartite graph. Then $\mathscr{P}_G$ is Gorenstein of index* 2 *if and only if G has a perfect matching and, for any subset $\emptyset \neq S \subsetneq V_1$ such that $G[S \cup \Gamma(S)]$ and $G[(V_1 \setminus S) \cup (V_2 \setminus \Gamma(S))]$ are connected, we have $|S| + 1 = |\Gamma(S)|$.*

Proof. From Proposition 7.1, $\mathscr{P}_G$ is Gorenstein of index 2 if and only if there exists $\alpha \in \mathbb{Z}^n$ such that the following hold:

(i) $\alpha(V_1) = \alpha(V_2)$;

(ii) If v is not a cut vertex, then $\alpha(v) = 1$;

(iii) If $\deg(v) \ge 2$, then $\alpha(v) = 1$;

(iv) If $G[S \cup \Gamma(S)]$ and $G[(V_1 \setminus S) \cup (V_2 \setminus \Gamma(S))]$ are connected for a subset $\emptyset \neq S \subsetneq V_1$, then $\alpha(S) - \alpha(\Gamma(S)) = -1$.

Two conditions (ii) and (iii) say that any vertex v in $V_1 \cup V_2$ satisfies $\alpha(v) = 1$ (i.e., $\alpha = (1, 1, \ldots, 1)$), since any vertex of degree one is not a cut vertex. Hence $\mathscr{P}_G$ is Gorenstein of index 2 if and only if

(a) $|V_1| = |V_2|$;

(b) If $G[S \cup \Gamma(S)]$ and $G[(V_1 \setminus S) \cup (V_2 \setminus \Gamma(S))]$ are connected for a subset $\emptyset \neq S \subsetneq V_1$, then $|S| - |\Gamma(S)| = -1$.

It is enough to show that (a) and (b) hold if and only if

(a') G has a perfect matching;

(b) If $G[S \cup \Gamma(S)]$ and $G[(V_1 \setminus S) \cup (V_2 \setminus \Gamma(S))]$ are connected for a subset $\emptyset \neq S \subsetneq V_1$, then $|S| - |\Gamma(S)| = -1$

hold. From Hall's marriage theorem, (a') $\Rightarrow$ (a). Hence we have "(a') and (b)" $\Rightarrow$ "(a) and (b)". Suppose that (a) and (b) hold. Then $\mathscr{P}_G$ is Gorenstein of index 2, and hence $\alpha = (1, \ldots, 1) \in 2(\mathscr{P}_G \setminus \partial \mathscr{P}_G)$. From Proposition 4.3, $|S| - |\Gamma(S)| \le 0$ for all $\emptyset \neq S \subset V_1$, and $|V_1| - |V_2| = 0$. From Hall's marriage theorem, G has a perfect matching. Thus (a') and (b) hold, as desired. □

Theorem 7.4. *Suppose that a connected bipartite graph G has a vertex v with* $\deg(v) \ge 2$ *such that v is not a cut vertex. Then the following conditions are equivalent*:

(i) *$\mathscr{P}_G$ is Gorenstein*;

(ii) $\mathscr{P}_G$ *is Gorenstein of index* 2;

(iii) *G has a perfect matching and, for any subset $\emptyset \neq S \subsetneq V_1$ such that $G[S \cup \Gamma(S)]$ and $G[(V_1 \setminus S) \cup (V_2 \setminus \Gamma(S))]$ are connected, we have $|S|+1 = |\Gamma(S)|$.*

Proof. From Proposition 7.3, (ii) $\Leftrightarrow$ (iii). It is obvious that (ii) $\Rightarrow$ (i). It remains prove that (i) $\Rightarrow$ (ii). Since α in Proposition 7.1 satisfies $\alpha(v) = \delta - 1 = 1$, we obtain $\delta = 2$. $\square$

On the other hand, the following is known.

Proposition 7.5 ([32, Theorem 2.1 (iii) (a')])**.** *Let G be a 2-connected bipartite graph. Then the edge polytope* $\mathrm{Ed}(G)$ *is Gorenstein if and only if G has a perfect matching and, for any subset $\emptyset \neq S \subset V_1$ such that $G[S \cup \Gamma(S)]$ is connected and that $G[(V_1 \setminus S) \cup (V_2 \setminus \Gamma(S))]$ is connected and has at least one edge, we have $|S|+1 = |\Gamma(S)|$.*

If G is a 2-connected bipartite graph, the conditions in Propositions 7.3 and 7.5 are equivalent. Since the statement is slightly different, we give a proof for the readers.

Corollary 7.6. *Let G be a 2-connected bipartite graph. Then $\mathscr{P}_G$ is Gorenstein if and only if the edge polytope* $\mathrm{Ed}(G)$ *of G is Gorenstein.*

Proof. Suppose that G is a 2-connected bipartite graph. Then $\deg(v) \geq 2$ for any vertex v of G.

Suppose that $\mathrm{Ed}(G)$ is Gorenstein. Assume that, for $\emptyset \neq S \subsetneq V_1$, $G[(V_1 \setminus S) \cup (V_2 \setminus \Gamma(S))]$ is a connected graph with no edges. Then $G[(V_1 \setminus S) \cup (V_2 \setminus \Gamma(S))]$ has exactly one vertex v. It then follows that $V_1 = S \cup \{v\}$ and $V_2 = \Gamma(S)$. Since G has a perfect matching, we have $|V_1| = |V_2|$. Thus, $|S|+1 = |V_1| = |V_2| = |\Gamma(S)|$ in this case. Thus, by Proposition 7.4, $\mathscr{P}_G$ is Gorenstein.

Suppose that $\mathscr{P}_G$ is Gorenstein. Assume that, for $\emptyset \neq S \subset V_1$, $G[(V_1 \setminus S) \cup (V_2 \setminus \Gamma(S))]$ is a connected graph. If $S \neq V_1$, then $|S|+1 = |\Gamma(S)|$ by Proposition 7.4. If $S = V_1$, then $\Gamma(S) = V_2$ and hence $G[(V_1 \setminus S) \cup (V_2 \setminus \Gamma(S))]$ has no vertices. Thus, by Proposition 7.5, $\mathrm{Ed}(G)$ is Gorenstein. $\square$

Theorem 4.7 follows from Theorem 7.4 and Corollary 7.6.

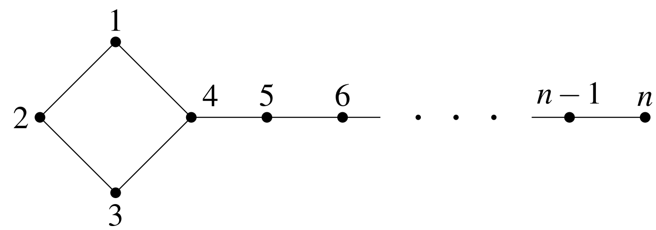
Remark 7.7. The conclusion of Corollary 7.6 is not true if G is not a 2-connected bipartite graph. There are many bipartite graphs G such that $\mathrm{Ed}(G)$ is Gorenstein and $\mathscr{P}_G$ is not Gorenstein.

(a) Let G be a bipartite pseudotree. Then the edge polytope $\mathrm{Ed}(G)$ of G is Gorenstein since the toric ring of $\mathrm{Ed}(G)$ is either isomorphic to a polynomial ring or a hypersurface.

(b) Let G be a bipartite pseudotree. Then $\mathscr{P}_G$ is Gorenstein if and only if G satisfies either (i) or (iv) in Theorem 4.8. For example, the perfectly matchable subgraph polytope of the bipartite pseudotree with the edge set

$$\{\{1,2\},\{2,3\},\{3,4\},\{1,4\}\} \cup \{\{4,5\},\{5,6\},\dots,\{n-1,n\}\}$$

is not Gorenstein.



In fact, $\{1,2,3,4\}$ is an even cycle of the graph, but $\deg(4) = 3 \neq \deg(1) = \deg(2) = \deg(3) = 2$. Thus G satisfies neither (i) nor (iv) in Theorem 4.8.

Proposition 7.8. *Let G be a 2-connected bipartite graph. Then we have the following.*

(a) If G is outerplanar, then $\mathscr{P}_G$ is Gorenstein.

(b) If G is 4-connected and planar, then $\mathscr{P}_G$ is not Gorenstein.

Proof. Let G be a 2-connected bipartite graph with n vertices.

(a) Suppose that G is outerplanar. Then G has an even cycle $(i_1, i_2, \ldots, i_n)$ of length n which corresponds to the outer face of G. Suppose that, for a subset $\emptyset \neq S \subsetneq V_1$, both $G[S \cup \Gamma(S)]$ and $G[(V_1 \setminus S) \cup (V_2 \setminus \Gamma(S))]$ are connected. We may assume that

$$S \cup \Gamma(S) = \{i_1, i_2, \ldots, i_{k'_1}\} \cup \{i_{k_2}, i_{k_2+1}, \ldots, i_{k'_2}\} \cup \cdots \cup \{i_{k_p}, i_{k_p+1}, \ldots, i_{k'_p}\},$$

where $k'_j + 1 < k_{j+1}$ for each $j = 1, 2, \ldots, p-1$. Since $S \neq V_1$, we may assume that $k'_p < n$.

Case 1. ($p = 1$) If i_1 (resp. $i_{k'_1}$) belongs to S, then i_n (resp. $i_{k'_1+1}$) belongs to $\Gamma(S)$. This is a contradiction. Hence i_1 and $i_{k'_1}$ belong to $\Gamma(S)$. Then $S = \{i_2, i_4, \ldots, i_{k'_1-1}\}$ and $\Gamma(S) = \{i_1, i_3, \ldots, i_{k'_1}\}$. Hence $|S| + 1 = |\Gamma(S)|$.

Case 2. ($p \geq 2$) Since $G[S \cup \Gamma(S)]$ is connected, there exists an edge $e_1 = \{i_\alpha, i_\beta\}$ of $G[S \cup \Gamma(S)]$, where $1 \leq \alpha \leq k'_1$ and $k_q \leq \beta \leq k'_q$ for some $2 \leq q \leq p$. On the other hand, since $G[(V_1 \setminus S) \cup (V_2 \setminus \Gamma(S))]$ is connected, there exists an edge $e_2 = \{i_\gamma, i_\delta\}$ of $G[(V_1 \setminus S) \cup (V_2 \setminus \Gamma(S))]$, where $k'_1 < \gamma < k_q$ and $k'_q < \delta \leq n$. Then e_1 and e_2 intersect in the drawing. This contradicts that G is outerplanar.

Thus, G satisfies the condition in Proposition 7.3, and hence $\mathscr{P}_G$ is Gorenstein.

(b) Suppose that G is 4-connected and planar. Let $S = \{v\}$, where $v \in V_1$ is a vertex of G. Since G is 4-connected, the degree of each vertex of G is greater than or equal to 4. Hence, we have $|\Gamma(S)| \geq 4 > |S| + 1$. In addition, $G[S \cup \Gamma(S)]$ is a star graph and hence connected. Since G is 4-connected and planar, it is known [9, Lemma 1] that $G[(V_1 \setminus S) \cup (V_2 \setminus \Gamma(S))]$ is connected. Hence, G does not satisfy the condition in Proposition 7.3. Thus, $\mathscr{P}_G$ is not Gorenstein. □

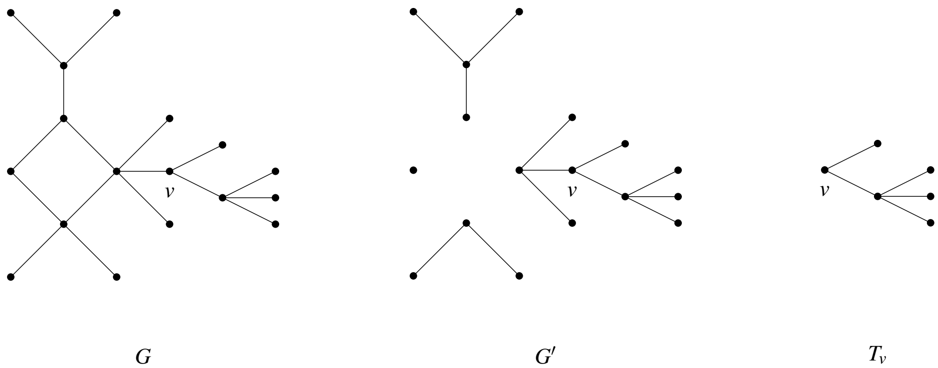


Figure 7.1: Graph G and the subtree which has v as the root

7.2 Pseudotrees with an even cycle

In this section, we give a proof of Theorem 4.8 for a pseudotree which has an even cycle.

Proof of Theorem 4.8 (when G has an even cycle C). Let $V = V_c \sqcup V_t \sqcup V_p$ be a partition of the vertex set V of G, where $V_c = \{v_1, \ldots, v_{2n}\}$ is the vertex set of the even cycle $C = (v_1, v_2, \ldots, v_{2n})$, $V_t = \{v \in V \setminus V_c : \deg(v) > 1\}$, and $V_p = \{v \in V \setminus V_c : \deg(v) = 1\}$. The graph G' obtained from G by deleting edges of C has $2n$ connected components. Each of these connected components of G' is a tree. We regard each tree as a rooted tree whose root is a vertex in C. Let T_v be the rooted subtree of such a rooted tree in G' whose root is v. See Figure 7.1.

("If") Suppose that graph G satisfies the condition that there exists an integer $\delta \geq 2$ such that $\deg(v) = \delta$ if $v \in V_c$ and $\deg(v) = \delta - 1$ if $v \in V_t$. By Proposition 7.1, it is enough to show that $\alpha \in \mathbb{Z}^n$, where

$$\alpha(v) = \begin{cases} \delta - 1 & \text{if } v \in V_c \cup V_t \\ 1 & \text{if } v \in V_p \end{cases}$$

satisfies conditions (i)–(iv) in Proposition 7.1.

(i) Note that, for a bipartite graph on the vertex set $V_1 \cup V_2$, $\sum_{v \in V_1} \deg(v) = \sum_{v \in V_2} \deg(v)$. Since $\alpha(v) = \deg(v)$ if $v \in V \setminus V_c$ and $\alpha(v) = \deg(v) - 1$ if $v \in V_c$, we have

$$\begin{aligned} \alpha(V_1) - \alpha(V_2) &= \left(\sum_{v \in V_1} \deg(v) - |V_c \cap V_1| \right) - \left(\sum_{v \in V_2} \deg(v) - |V_c \cap V_2| \right) \\ &= |V_c \cap V_2| - |V_c \cap V_1| = n - n = 0. \end{aligned}$$

(ii) Note that v is not a cut vertex if and only if either (a) $\deg(v) = 1$ or (b) $v \in V_c$ and $\deg(v) = 2$ (then $\delta = 2$ and $G = C$). In both cases, we have $\alpha(v) = 1$.

(iii) If $\deg(v) \geq 2$, then $v \in V_c \cup V_t$ and hence we have $\alpha(v) = \delta - 1$.

(iv) Suppose that $G[S \cup \Gamma(S)]$ and $G[(V_1 \setminus S) \cup (V_2 \setminus \Gamma(S))]$ are connected. Then S satisfies one of the following:

Case 1. ($V_c \cap S = \emptyset$) Since both $G[S \cup \Gamma(S)]$ and $G[(V_1 \setminus S) \cup (V_2 \setminus \Gamma(S))]$ are connected, $G[S \cup \Gamma(S)]$ is T_v, where $v \in \Gamma(S)$. Since S and $\Gamma(S)$ give a partition of the vertex set of bipartite graph $G[S \cup \Gamma(S)]$, the sum of the degree sequence of S is equal to that of $\Gamma(S)$ in $G[S \cup \Gamma(S)]$.

Case 1.1. ($v \in V_t$) Then $\deg_G(v) = \deg_{G[S \cup \Gamma(S)]}(v) + 1$. Since $(S \cup \Gamma(S)) \cap V_c = \emptyset$,

$$\begin{aligned} \alpha(S) - \alpha(\Gamma(S)) &= \sum_{v' \in S} \deg_G(v') - \sum_{v' \in \Gamma(S)} \deg_G(v') \\ &= \sum_{v' \in S} \deg_{G[S \cup \Gamma(S)]}(v') - (1 + \sum_{v' \in \Gamma(S)} \deg_{G[S \cup \Gamma(S)]}(v')) \\ &= -1. \end{aligned}$$

Case 1.2. ($v \in V_c$) Then $\deg_G(v) = \deg_{G[S \cup \Gamma(S)]}(v) + 2$. Since $(S \cup \Gamma(S)) \cap V_c = \{v\}$,

$$\begin{aligned} \alpha(S) - \alpha(\Gamma(S)) &= \sum_{v' \in S} \deg_G(v') - (-1 + \sum_{v' \in \Gamma(S)} \deg_G(v')) \\ &= \sum_{v' \in S} \deg_{G[S \cup \Gamma(S)]}(v') - (-1 + 2 + \sum_{v' \in \Gamma(S)} \deg_{G[S \cup \Gamma(S)]}(v')) \\ &= -1. \end{aligned}$$

Case 2. ($V_c \subset (S \cup \Gamma(S))$) Let $S' = V_2 \setminus \Gamma(S)$. Since $G[(V_1 \setminus S) \cup (V_2 \setminus \Gamma(S))]$ is connected, we have $\Gamma(S') = V_1 \setminus S$. By the condition (i) and Remark 4.5, then we have $\alpha(S) - \alpha(\Gamma(S)) = \alpha(S') - \alpha(\Gamma(S'))$. Note that $V_c \cap (S' \cup \Gamma(S)) = \emptyset$. By the argument of Case 1.1, we have $\alpha(S) - \alpha(\Gamma(S)) = \alpha(S') - \alpha(\Gamma(S')) = -1$.

Case 3. (otherwise) Since $G[S \cup \Gamma(S)]$ is connected, we may assume that

$$V_c \cap S = \{v_2, v_4, \ldots, v_{2m}\} \text{ and } V_c \cap \Gamma(S) = \{v_1, v_3, \ldots, v_{2m+1}\}$$

by rearranging indices if necessary.

Case 3.1. ($\deg(v_i) = 2$ for some i) Then $\delta = 2$ and $G = C$. Since $\alpha(v_j) = 1$ for $j = 1, 2, \ldots, 2m+1$, we have $\alpha(S) - \alpha(\Gamma(S)) = m - (m+1) = -1$.

Case 3.2. ($\deg(v_i) \geq 3$ for all i) Then $G[S \cup \Gamma(S)]$ is a graph whose edge set is

$$\{\{v_1, v_2\}, \ldots, \{v_{2m}, v_{2m+1}\}\} \cup T_{v_1} \cup T_{v_2} \cup \cdots \cup T_{v_{2m+1}}.$$

By the argument in Case 1.2, $\alpha(S \cap T_{v_i}) - \alpha(\Gamma(S) \cap T_{v_i}) = (-1)^i$. Then $\alpha(S) - \alpha(\Gamma(S)) = \sum_{i=1}^{2m+1} (-1)^i = -1$.

In any case, we have $\alpha(S)-\alpha(\Gamma(S))=-1$.

Hence, by Proposition 7.1, $\mathscr{P}_G$ is Gorenstein.

("Only if") Suppose that $\mathscr{P}_G$ is Gorenstein. Since any even cycle satisfies condition (iv) in Theorem 4.8, we may assume that G is not an even cycle. Then we have $V_p \neq \emptyset$. By Proposition 7.1, there exist δ and α satisfying conditions (i)–(iv). By conditions (ii) and (iii), we have

$$\alpha(v)=\begin{cases}\delta-1 & \text{if } v\in V_c\cup V_t,\\ 1 & \text{if } v\in V_p.\end{cases}$$

Case 1. (V_c has a vertex which is not a cut vertex) Since every vertex v in V_c satisfies $\deg(v)\geq 2$, we obtain $\delta-1=1$ and hence $\delta=2$, $\alpha=(1,\ldots,1)$. Since G is not an even cycle, there exist $S\subset V_p$ and $v\in V_t\cup V_c$ such that $\Gamma(S)=\{v\}$. Since $G[S\cup\Gamma(S)]$ is K_2 or star graph, $\alpha(S)-\alpha(\Gamma(S))\geq 0$. Since both $G[S\cup\Gamma(S)]$ and $G[(V_1\setminus S)\cup(V_2\setminus\Gamma(S))]$ are connected, this contradicts to condition (iv).

Case 2. (any vertex in V_c is a cut vertex) Let $v\in V_t\cup V_c$. Then v is a cut vertex. Let S be a subset of V_1 (or V_2) such that $G[S\cup\Gamma(S)]$ is a subtree T_v which has $v\in\Gamma(S)$ as a root. Let $v'\in S$ be a child of v, i.e., $\{v,v'\}$ is an edge of T_v. If $\deg_G(v')\geq 2$, then there exists a subset $S_{v'}\subset\Gamma(S)$ such that $G[S_{v'}\cup\Gamma(S_{v'})]$ is a rooted subtree of $G[S\cup\Gamma(S)]$ with root $v'\in\Gamma(S_{v'})$. By condition (iv), $\alpha(\Gamma(S_{v'}))-\alpha(S_{v'})=1$. If $\deg(v')=1$, then $\alpha(v')=1$. Since $T_v\setminus\{v\}$ is a disjoint union of rooted subtrees in $\{T_{v'} : v' \text{ is a child of } v\}$,

$$\begin{aligned} S &= \bigcup_{v' \text{ is a child of } v}\Gamma(S_{v'})\\ \Gamma(S) &= \{v\}\cup\bigcup_{v' \text{ is a child of } v} S_{v'}\end{aligned}$$

are partitions of S and $\Gamma(S)$, respectively. Hence

$$\begin{aligned} -1 &= \alpha(S)-\alpha(\Gamma(S))\\ &= -\alpha(v)+\sum_{v' \text{ is a child of } v}(\alpha(\Gamma(S_{v'}))-\alpha(S_{v'}))\\ &= -(\delta-1)+\sum_{v' \text{ is a child of } v}1\\ &= -\delta+1+\deg_{G[S\cup\Gamma(S)]}(v).\end{aligned}$$

If $v\in V_c$, then

$$\deg_G(v)=\deg_{G[S\cup\Gamma(S)]}(v)+2=\delta.$$

If $v\in V_t$, then

$$\deg_G(v)=\deg_{G[S\cup\Gamma(S)]}(v)+1=\delta-1.$$

□

Example 7.9. The perfectly matchable subgraph polytope $\mathscr{P}_G$ of the graph G in Figure 7.2 is Gorenstein.

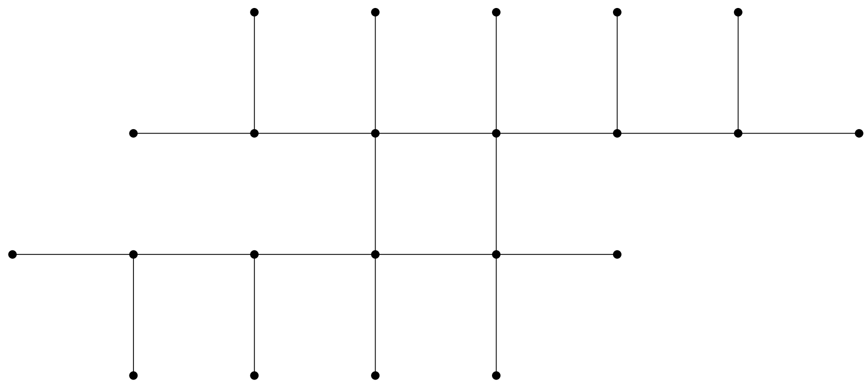


Figure 7.2: $\deg(v_c) = 4$ and $\deg(v_t) = 3$

7.3 Pseudotrees without even cycles

In this section, we give a proof of Theorem 4.8 for a pseudotree which has no even cycles.

A graph G is said to be *h-perfect* if the stable set polytope $\mathrm{Stab}(G)$ is defined by the following constraints:

$$x(v) \geq 0 \quad \text{for any } v \in V(G), \tag{7.5}$$
$$x(K) \leq 1 \quad \text{for any maximal clique } K \text{ in } G, \tag{7.6}$$
$$x(C) \leq n \quad \text{for any induced odd cycle } C \text{ in } G \text{ of length } 2n+1 \geq 5. \tag{7.7}$$

In particular, any perfect graph is h-perfect. An *odd subdivision* of a graph G is a graph obtained by replacing each edge of G by a path of odd length. Let $C_5 + e$ be the graph on the vertex set $\{1,\dots,5\}$ and the edge set $\{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{1,5\},\{1,3\}\}$. It is known [4, Theorem 5] that $L(G)$ is h-perfect if and only if G has no odd subdivision of $C_5 + e$. Since any pseudotree has at most one cycle, we have the following immediately.

Lemma 7.10. *Let G be a pseudotree. Then $L(G)$ is h-perfect.*

Let K be a field. The *Ehrhart ring* of a lattice polytope $P \subset \mathbb{R}^n$ is

$$K[\mathbf{x}^{\alpha} s^m : \alpha \in mP \cap \mathbb{Z}^n, m \in \mathbb{Z}_{\geq 0}] \subset K[x_1^{\pm},\dots,x_n^{\pm},s].$$

It is known that the Ehrhart ring of P coincides with the toric ring of P if and only if P has IDP. For the Ehrhart ring of $\mathrm{Stab}(G)$, the following fact is known.

Proposition 7.11 ([24, Theorem 3.8])**.** *Let G be an h-perfect graph. Then the Ehrhart ring of* $\mathrm{Stab}(G)$ *is Gorenstein if and only if all maximal cliques of G have the same cardinality (say ω), and that G satisfies one of the following conditions*:

(i) $\omega = 1$;

(ii) $\omega = 2$ *and G has no induced odd cycles of length* ≥ 7;

(iii) $\omega \geq 3$ *and G has no induced odd cycles of length* ≥ 5.

Recall that a *star* graph is a complete bipartite graph $K_{1,n}$. Then $L(K_{1,n})$ is K_n. On the other hand, we have $L(C_3) = K_3$. It is known that, in general, each clique in $L(G)$ corresponds to a star or to a triangle in G. Note that an induced cycle in $L(G)$ of length ≥ 4 corresponds to a (not necessarily induced) cycle in G of the same length. From these facts, we have the following.

Proof of Theorem 4.8 (when G has no even cycles). From Proposition 5.7, $K[\mathscr{P}_G]$ is isomorphic to $K[\mathrm{Stab}(L(G))]$. Hence $K[\mathscr{P}_G]$ is Gorenstein if and only if $K[\mathrm{Stab}(L(G))]$ is Gorenstein. Moreover, from Proposition 5.8, $\mathrm{Stab}(L(G))$ is normal. Since the lattice spanned by $\mathrm{Stab}(L(G)) \cap \mathbb{Z}^m$ is equal to $\mathbb{Z}^m$, $\mathrm{Stab}(L(G))$ has IDP. Hence, the toric ring of $\mathrm{Stab}(L(G))$ coincides with the Ehrhart ring of $\mathrm{Stab}(L(G))$. Thus, $K[\mathscr{P}_G]$ is Gorenstein if and only if the Ehrhart ring of $\mathrm{Stab}(L(G))$ is Gorenstein.

From Lemma 7.10, $L(G)$ is h-perfect. Hence, by Proposition 7.11, the Ehrhart ring of $\mathrm{Stab}(L(G))$ is Gorenstein if and only if $L(G)$ satisfies one of conditions (i)–(iii) in Proposition 7.11. These conditions are equivalent to the following, respectively:

(i) $L(G)$ has no edges, i.e., G is K_1 or K_2.

(ii) G is either a path of length ≥ 2 or C_5.

(iii) G is either a bidegreed tree which is not a path, or G has a triangle C where $\deg(v) \in \{2,3\}$ if $v \in V(C)$, and $\deg(v) \in \{1,3\}$ if $v \in V \setminus V(C)$.

Thus, (i)–(iii) above hold if and only if $K[\mathscr{P}_G]$ is Gorenstein. □

7.4 Complete multipartite graphs

Proposition 7.12. *Let G be a complete bipartite graph $K_{p,q}$ ($p \leq q$). Then $K[\mathscr{P}_G]$ is Gorenstein (equivalently, $\mathscr{P}_G$ is Gorenstein) if and only if either $p = 1$ or $p = q$.*

Proof. If $p = 1$, then G is a star graph. By Theorem 4.8, $\mathscr{P}_G$ is Gorenstein. Let $1 < p \leq q$. Then every vertex v of G is not a cut vertex and satisfies $\deg(v) \geq 2$. By Theorem 7.4, $\mathscr{P}_G$ is not Gorenstein if $p \neq q$ since G has no perfect matchings. If $p = q$, then G has a perfect matching. Since G is complete bipartite, $\Gamma(S) = V_2$ for $\emptyset \neq S \subsetneq V_1$. Hence $G[S \cup \Gamma(S)]$ and $G[(V_1 \setminus S) \cup (V_2 \setminus \Gamma(S))]$ are connected if and only if $S = V_1 \setminus \{v\}$ for some $v \in V_1$. If $S = V_1 \setminus \{v\}$, then $p = |S| + 1 = |\Gamma(S)|$. By Proposition 7.3, $\mathscr{P}_G$ is Gorenstein. □

Let $\mathscr{P} \subset \mathbb{R}^n$ be a lattice polytope and let $\alpha \in \mathbb{Z}^n \setminus \mathrm{aff}(\mathscr{P})$. Then the convex hull of $\mathscr{P} \cup \{\alpha\}$ is called the *pyramid* over $\mathscr{P}$ with apex α. In general, if a lattice polytope $\mathscr{P}$ is a pyramid over a lattice polytope $\mathscr{P}'$, then $K[\mathscr{P}]$ is isomorphic to a polynomial ring in one variable over $K[\mathscr{P}']$.

Proposition 7.13. *Let G be a complete multipartite graph $K_{1,1,q}$ with $q \geq 1$. Then $K[\mathscr{P}_G]$ is Gorenstein if and only if $q \leq 2$.*

Proof. Let $G = K_{1,1,q}$ on the vertex set $\{1\} \cup \{2\} \cup \{3,4,\ldots,q+2\}$ and $G' = K_{2,q}$ on the vertex set $\{1,2\} \cup \{3,4,\ldots,q+2\}$. For $S = \{3,4,\ldots,q+2\}$, since S is a stable set in G and since the graph obtained from $G[S \cup \Gamma(S)]$ by deleting all edges with both ends in $\Gamma(S)$ is the connected graph G',

$$x(S) - x(\Gamma(S)) = x_3 + x_4 + \cdots + x_{q+2} - (x_1 + x_2) \leq 0$$

is facet-inducing for $\mathscr{P}_G$. In addition, the facet is $\mathscr{P}_{G'}$. It then follows that $\mathscr{P}_G$ is a pyramid over $\mathscr{P}_{G'}$ with apex $\mathbf{e}_1 + \mathbf{e}_2$. Hence the toric ring $K[\mathscr{P}_G]$ is isomorphic to a polynomial ring in one variable over $K[\mathscr{P}_{G'}]$. Thus $K[\mathscr{P}_G]$ is Gorenstein if and only if $K[\mathscr{P}_{G'}]$ is Gorenstein. From Proposition 7.12, $K[\mathscr{P}_{G'}]$ is Gorenstein if and only if $q \leq 2$. □

Let G be a simple graph on the vertex set $V = \{1,2,...,n\}$ and the edge set E. Given $S \subset V$, the cut semimetric on G induced by S is the (0,1) vector $\delta_G(S) = (d_{ij} : \{i,j\} \in E) \in \mathbb{R}^E$, where

$$d_{ij} = \begin{cases} 1 \text{ if } |S \cap \{i,j\}| = 1 \\ 0 \text{ otherwise} \end{cases}$$

for each $\{i,j\} \in E$. In particular, $\delta_G(\emptyset) = 0$. The *cut polytope*$\mathrm{Cut}^{\square}(G)$ of G is the convex hull of $\{\delta_G(S) : S \subset V\} \subset \mathbb{Z}^E$.

Proposition 7.14. *Let G be a complete graph K_n. Then $K[\mathscr{P}_G]$ is Gorenstein if and only if $n \leq 4$.*

Proof. By Theorem 4.8, $K[\mathscr{P}_G]$ is Gorenstein if $n \leq 3$.

Let $n \geq 4$. By the definition of perfectly matchable subgraphs, we have

$$\mathscr{W}(G) = \{S \subset V : |S| \equiv 0 \ (\mathrm{mod}\ 2)\},$$

since G is complete. Then $\mathscr{P}_G$ is the cut polytope of a cycle C_n of length n (see [20]) which is normal. It is known [28, Theorem 3.4] that the toric ring of the cut polytope of a graph H is normal and Gorenstein if and only if H has no K_5-minor and satisfies one of the following:

(i) H is a bipartite graph without induced cycle of length ≥ 6;

(ii) H is a bridgeless chordal graph.

Hence, the toric ring of the cut polytope of C_n is Gorenstein if and only if $n \leq 4$. □

It is an interesting problem to characterize complete multipartite graphs G such that $K[\mathscr{P}_G]$ is Gorenstein. However, $\mathscr{P}_G$ does not have IDP if G is not bipartite. In addition, the normality of $\mathscr{P}_G$ is unknown except for $K_{p,q}$, $K_{1,1,q}$, and K_n.

Appendix A

Source code

In this chapter we write source codes to compute the toric ideal of stable set polytope and perfectly matchable subgraph polytope for CoCoA.

At first, the following source code can compute the stable set from finite graph.

Listing A.1: Stable set

```
Define Stable(E,n)
  A:=[];
  B:=[];
  S:=subsets(1..n);
  for i:=1 to len(S) do
    for j:=1 to len(E) do
      If len(intersection(E[j],S[i]))=2 then append(ref A, (S[i]));
      else append(ref B, (S[i]));
      endif;
             endfor;
             endfor;
             return MakeSet(diff(B,A))
enddefine; -- Stable
```

Next, following source code can compute sets induce perfectly matchable subgraph polytope from finite graph.

Listing A.2: Perfectly matchable subgraph

```
Define tuika(A,B)
  If Len(intersection(A,B))=0 Then Return sorted(flatten([A,B]));
  else return []
  EndIf;
EndDefine; --tuika

define Match(E)
  F:=[];
  for i:=1 to len(E) do
    for j:=1 to len(E) do
      append(ref F ,(tuika(E[i],E[j])))
    endfor;
```

```
  endfor
  return MakeSet(concat(E,F));
enddefine; -- Match
```

Finally, we introduce the source code which generate (0,1)-polytope from finite sets.

Listing A.3: Make triangulation

```
Define henkan(A,n)
  zeero:=[0| i in 1..n+1];
  foreach i in A do
  zeero[i]:=zeero[i]+1;
  endforeach;
  zeero[n+1]:=1;
  return zeero;
enddefine; -- henkan


Define gyoretu(E,n)
  F:=[];
  for i:=1 to len(E) do
    append(ref F, (henkan(E[i],n)))
  endfor;
  return transposed(Mat(MakeSet(F)))
enddefine; -- gyoretu
```

By using them, we can compute the algebraic properties of toric rings of stable set polytopes and perfectly matchable subgraph polytopes.

Example A.1. We will compute toric ideal, its minimal generators and Hilbert series of $K[\mathrm{Stab}(C_6)]$.

```
/**/ E:=[[1,2],[2,3],[3,4],[4,5],[5,6],[1,6]];
/**/ S:=Stable(E,6);
/**/ S;
[[], [6], [5], [4], [4, 6], [3], [3, 6], [3, 5], [2], [2, 6],
    [2, 5], [2, 4], [2, 4, 6], [1], [1, 5], [1, 4], [1, 3], [1,
     3, 5]]
/**/ n:=len(S);
/**/ Use R::=QQ[x[1..n]],DegLex ;
/**/ gyoretu(S,6);
matrix(QQ,
 [[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1],
  [0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0],
  [0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1],
  [0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 1, 0, 0],
  [0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1],
  [0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0],
  [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]])
/**/ I:=toric(gyoretu(S,6));
/**/ I;
ideal(x[1]*x[8] -x[3]*x[6], x[1]*x[11] -x[3]*x[9], x[1]*x[15] -x
    [3]*x[14], x[1]*x[5] -x[2]*x[4], x[1]*x[12] -x[4]*x[9], x
```

```
    [1]*x[10] -x[2]*x[9], x[1]*x[16] -x[4]*x[14], x[1]*x[17] -x
    [6]*x[14], -x[1]*x[7] +x[2]*x[6], x[3]*x[12] -x[4]*x[11], -x
    [9]*x[15] +x[11]*x[14], -x[4]*x[17] +x[6]*x[16], x[1]*x[13]
    -x[5]*x[9], x[6]*x[11] -x[8]*x[9], -x[9]*x[16] +x[12]*x[14],
     x[2]*x[17] -x[7]*x[14], x[3]*x[17] -x[6]*x[15], x[6]*x[15]
    -x[8]*x[14], x[3]*x[16] -x[4]*x[15], x[4]*x[10] -x[5]*x[9],
    -x[1]*x[18] +x[8]*x[14], x[2]*x[12] -x[5]*x[9], x[2]*x[8] -x
    [3]*x[7], x[6]*x[10] -x[7]*x[9], x[2]*x[11] -x[3]*x[10], x
    [2]*x[16] -x[5]*x[14], -x[4]*x[7] +x[5]*x[6], -x[1]^2*x[18]
    +x[3]*x[6]*x[14], x[11]*x[16] -x[12]*x[15], x[5]*x[17] -x
    [7]*x[16], -x[7]*x[11] +x[8]*x[10], -x[9]*x[18] +x[11]*x
    [17], -x[4]*x[13] +x[5]*x[12], x[10]*x[16] -x[13]*x[14], -x
    [4]*x[18] +x[8]*x[16], -x[3]*x[18] +x[8]*x[15], x[6]*x[13] -
    x[7]*x[12], x[3]*x[13] -x[5]*x[11], -x[14]*x[18] +x[15]*x
    [17], -x[2]*x[18] +x[7]*x[15], x[2]*x[13] -x[5]*x[10], -x
    [9]*x[13] +x[10]*x[12], -x[6]*x[18] +x[8]*x[17], -x[1]*x[2]*
    x[18] +x[3]*x[7]*x[14], -x[2]*x[9]*x[15] +x[3]*x[10]*x[14],
    -x[2]*x[4]*x[15] +x[3]*x[5]*x[14], -x[4]*x[7]*x[11] +x[5]*x
    [8]*x[9], x[4]*x[13]*x[14] -x[5]*x[9]*x[16], x[4]*x[13]*x
    [17] -x[7]*x[12]*x[16], -x[7]*x[11]*x[12] +x[8]*x[9]*x[13])
/**/ MinGens(I);
[-x[14]*x[18] +x[15]*x[17], x[11]*x[16] -x[12]*x[15], x[10]*x
    [16] -x[13]*x[14], -x[9]*x[18] +x[11]*x[17], -x[9]*x[16] +x
    [12]*x[14], -x[9]*x[15] +x[11]*x[14], -x[9]*x[13] +x[10]*x
    [12], -x[7]*x[11] +x[8]*x[10], -x[6]*x[18] +x[8]*x[17], x
    [6]*x[15] -x[8]*x[14], x[6]*x[13] -x[7]*x[12], x[6]*x[11] -x
    [8]*x[9], x[6]*x[10] -x[7]*x[9], x[5]*x[17] -x[7]*x[16], -x
    [4]*x[18] +x[8]*x[16], -x[4]*x[17] +x[6]*x[16], -x[4]*x[13]
    +x[5]*x[12], x[4]*x[10] -x[5]*x[9], -x[4]*x[7] +x[5]*x[6], -
    x[3]*x[18] +x[8]*x[15], x[3]*x[17] -x[8]*x[14], x[3]*x[16] -
    x[4]*x[15], x[3]*x[13] -x[5]*x[11], x[3]*x[12] -x[4]*x[11],
    -x[2]*x[18] +x[7]*x[15], x[2]*x[17] -x[7]*x[14], x[2]*x[16]
    -x[5]*x[14], x[2]*x[13] -x[5]*x[10], x[2]*x[12] -x[5]*x[9],
    x[2]*x[11] -x[3]*x[10], x[2]*x[8] -x[3]*x[7], -x[1]*x[18] +x
    [8]*x[14], x[1]*x[17] -x[6]*x[14], x[1]*x[16] -x[4]*x[14], x
    [1]*x[15] -x[3]*x[14], x[1]*x[13] -x[5]*x[9], x[1]*x[12] -x
    [4]*x[9], x[1]*x[11] -x[3]*x[9], x[1]*x[10] -x[2]*x[9], x
    [1]*x[8] -x[3]*x[6], -x[1]*x[7] +x[2]*x[6], x[1]*x[5] -x[2]*
    x[4]]
/**/ HilbertSeries(R/I);
(1 + 11*t + 24*t^2 + 11*t^3 + t^4) / (1-t)^7
```

Example A.2. Let G be a cycle of length 6. We will compute $K[\mathscr{P}_G]$. The edge set of C_6 is $[1,2],[2,3],[3,4],[4,5],[5,6],[1,6]$.

```
/**/ E:=[[1,2],[2,3],[3,4],[4,5],[5,6],[1,6]];
/**/ M:=Match(Match(Match((E))));
M;
[[1, 2], [2, 3], [3, 4], [4, 5], [5, 6], [1, 6], [], [1, 2, 3,
    4], [1, 2, 4, 5], [1, 2, 5, 6], [2, 3, 4, 5], [2, 3, 5, 6],
     [1, 2, 3, 6], [3, 4, 5, 6], [1, 3, 4, 6], [1, 4, 5, 6],
    [1, 2, 3, 4, 5, 6]]
```

```
/**/ n:=len(M);
/**/ Use R::=QQ[x[1..n]],DegLex ;
/**/ gyoretu(M,6);
matrix(QQ,
 [[1, 0, 0, 0, 0, 1, 0, 1, 1, 1, 0, 0, 1, 0, 1, 1, 1],
  [1, 1, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 0, 0, 0, 1],
  [0, 1, 1, 0, 0, 0, 0, 1, 0, 0, 1, 1, 1, 1, 1, 0, 1],
  [0, 0, 1, 1, 0, 0, 0, 1, 1, 0, 1, 0, 0, 1, 1, 1, 1],
  [0, 0, 0, 1, 1, 0, 0, 0, 1, 1, 1, 1, 0, 1, 0, 1, 1],
  [0, 0, 0, 0, 1, 1, 0, 0, 0, 1, 0, 1, 1, 1, 1, 1, 1],
  [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]])
/**/ I:=toric(gyoretu(M,6));
/**/ I;
ideal(x[1]*x[5] -x[7]*x[10], -x[1]*x[3] +x[7]*x[8], -x[1]*x[4] +
    x[7]*x[9], -x[2]*x[5] +x[7]*x[12], -x[4]*x[6] +x[7]*x[16], -
    x[2]*x[4] +x[7]*x[11], -x[2]*x[6] +x[7]*x[13], -x[3]*x[6] +x
    [7]*x[15], -x[3]*x[5] +x[7]*x[14], x[5]*x[13] -x[6]*x[12], x
    [5]*x[15] -x[6]*x[14], x[2]*x[16] -x[6]*x[11], -x[3]*x[10] +
    x[6]*x[11], -x[3]*x[10] +x[4]*x[13], -x[3]*x[10] +x[7]*x
    [17], x[1]*x[14] -x[3]*x[10], -x[3]*x[9] +x[4]*x[8], x[1]*x
    [12] -x[2]*x[10], -x[3]*x[10] +x[5]*x[8], x[1]*x[15] -x[6]*x
    [8], x[4]*x[12] -x[5]*x[11], x[2]*x[14] -x[3]*x[12], -x[4]*x
    [10] +x[5]*x[9], x[1]*x[11] -x[2]*x[9], x[1]*x[16] -x[6]*x
    [9], -x[3]*x[16] +x[4]*x[15], x[2]*x[15] -x[3]*x[13], x[2]*x
    [4]*x[6] -x[3]*x[7]*x[10], -x[3]*x[17] +x[8]*x[14], x[2]*x
    [17] -x[11]*x[13], x[6]*x[17] -x[13]*x[16], x[4]*x[17] -x
    [11]*x[16], -x[3]*x[17] +x[11]*x[15], x[1]*x[17] -x[8]*x
    [10], -x[9]*x[14] +x[11]*x[16], x[5]*x[17] -x[10]*x[14], -x
    [10]*x[14] +x[12]*x[16], -x[8]*x[12] +x[11]*x[13], x[9]*x
    [12] -x[10]*x[11], -x[8]*x[10] +x[9]*x[13], -x[10]*x[15] +x
    [13]*x[16], x[12]*x[15] -x[13]*x[14], x[8]*x[16] -x[9]*x
    [15], -x[1]*x[3]*x[10] +x[2]*x[6]*x[9], -x[3]*x[5]*x[16] +x
    [4]*x[6]*x[14], -x[1]*x[3]*x[13] +x[2]*x[6]*x[8], -x[3]*x
    [10]*x[13] +x[6]*x[8]*x[12], -x[3]*x[10]*x[16] +x[6]*x[9]*x
    [14], -x[3]*x[16]*x[17] +x[9]*x[14]*x[15])
/**/ MinGens(I);
[x[12]*x[15] -x[13]*x[14], -x[10]*x[15] +x[13]*x[16], -x[10]*x
    [14] +x[12]*x[16], -x[9]*x[14] +x[11]*x[16], x[9]*x[12] -x
    [10]*x[11], x[8]*x[16] -x[9]*x[15], -x[8]*x[12] +x[11]*x
    [13], -x[8]*x[10] +x[9]*x[13], x[6]*x[17] -x[13]*x[16], x
    [5]*x[17] -x[12]*x[16], x[5]*x[15] -x[6]*x[14], x[5]*x[13] -
    x[6]*x[12], x[4]*x[17] -x[11]*x[16], x[4]*x[12] -x[5]*x[11],
     -x[4]*x[10] +x[5]*x[9], -x[4]*x[6] +x[7]*x[16], -x[3]*x[17]
     +x[11]*x[15], -x[8]*x[14] +x[11]*x[15], -x[3]*x[16] +x[4]*x
    [15], -x[3]*x[10] +x[7]*x[17], x[4]*x[13] -x[7]*x[17], -x
    [6]*x[11] +x[7]*x[17], x[5]*x[8] -x[7]*x[17], -x[3]*x[9] +x
    [4]*x[8], -x[3]*x[6] +x[7]*x[15], -x[3]*x[5] +x[7]*x[14], x
    [2]*x[17] -x[11]*x[13], x[2]*x[16] -x[7]*x[17], x[2]*x[15] -
    x[3]*x[13], x[2]*x[14] -x[3]*x[12], -x[2]*x[6] +x[7]*x[13],
    -x[2]*x[5] +x[7]*x[12], -x[2]*x[4] +x[7]*x[11], x[1]*x[17] -
    x[9]*x[13], x[1]*x[16] -x[6]*x[9], x[1]*x[15] -x[6]*x[8], x
    [1]*x[14] -x[7]*x[17], x[1]*x[12] -x[2]*x[10], x[1]*x[11] -x
```

```
    [2]*x[9], x[1]*x[5] -x[7]*x[10], -x[1]*x[4] +x[7]*x[9], -x
    [1]*x[3] +x[7]*x[8]]
/**/ HilbertSeries(R/I);
(1 + 11*t + 24*t^2 + 11*t^3 + t^4) / (1-t)^6
```

Example A.3. Let G be a C_5. By Theorem 4.8, $\mathscr{P}_G$ is Gorenstein.

```
/**/ E:=[[1,2],[2,3],[3,4],[4,5],[1,5]];
/**/ M:=Match(Match(Match((E))));
/**/ n:=len(M);
/**/ gyoretu(M,5);
matrix(QQ,
 [[1, 0, 0, 0, 1, 0, 1, 1, 0, 1, 1],
  [1, 1, 0, 0, 0, 0, 1, 1, 1, 1, 0],
  [0, 1, 1, 0, 0, 0, 1, 0, 1, 1, 1],
  [0, 0, 1, 1, 0, 0, 1, 1, 1, 0, 1],
  [0, 0, 0, 1, 1, 0, 0, 1, 1, 1, 1],
  [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]])
/**/ Use R::=QQ[x[1..n]],DegRevLex ;
/**/ I:=toric(gyoretu(M,5));
/**/ HilbertSeries(R/I);
(1 + 5*t + 5*t^2 + t^3) / (1-t)^6
```

We computed the toric ideals of perfectly matchable subgraph polytopes for a lot of graphs. However, we cannot find the graph whose toric ideal of perfectly matchable subgraph polytope is not generated by quadratic binomials.

Conjecture A.4. Let G be a graph. The toric ideal $I_{\mathscr{P}_G}$ is generated by quadratic binomials.

If above conjecture is true, we should study the following conjecture.

Conjecture A.5. Let G be a graph. There exists a monomial order such that Gröbner basis of $I_{\mathscr{P}_G}$ is quadratic.

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