関西学院大学審査 博士学位論文

The (k, a)-Generalized Fourier Analysis

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Contents

1 Introduction

Dunkl theory is a far-reaching generalization of Euclidean Fourier analysis associated with root system with a rich structure parallel to ordinary Fourier analysis. The study of Dunkl theory originates from a generalization of spherical harmonics, in which the finite reflection groups G play the role of orthogonal group $O(N)$ in the classical theory of spherical harmonics. The Lebesgue measure dx, which is invariant under $O(N)$, is substituted by the Dunkl weight measure $dm_k(x) = h_k(x)dx$ which is invariant under the finite reflection group G and parameterized by a multiplicity function k, where $h_k(x) = \prod_{\alpha \in R} |\langle \alpha,\ x\rangle|^{k(\alpha)}$. The Dunkl operator T_i (see [18]) was constructed in such a way that the intersection of space of the homogeneous polynomials P_m of degree m with the kernel of the corresponding Laplacian $\triangle_k = \sum_{j=1}^{N} T_j^2$ is orthogonal to that of lower degree with respect to the Dunkl weight measure dm_k. And the restrictions of the spaces $\mathcal{H}_k^m\left(\mathbb{R}^N\right) := P_m \cap ker\triangle_k,\ m = 0, 1, \cdots$ to the unit sphere $\mathbb{S}^{N-1}$ are called spherical h-harmonics. The Dunkl operators commute pairwise and they are in substitute of the ordinary partial derivatives in classical analysis. The joint eigenfunctions of Dunkl operators take the place of the exponential functions in classical Fourier transform. The Dunkl transform (see [20]) was then defined correspondingly and has many similar properties with Fourier transform. The discovery of Dunkl operators also gave an explicit expression of the radial part of the Laplacian operator on a flat symmetric space unintentionally. Moreover, Dunkl theory has extensive application in algebra, probability theory and mathematical physics. This theory has drawn considerable attention and there have been a lot of works on Dunkl's analysis in the last thirty years.

More recently, S. Ben Saïd, T. Kobayashi and B. Ørsted [8] gave a further far-reaching generalization of Dunkl theory by introducing a parameter $a > 0$ arisen from the "interpolation" of the two $\mathfrak{sl}(2, \mathbb{R})$ actions on the Weil representation of the metaplectic group $Mp(N, \mathbb{R})$ and the minimal unitary representation of the conformal group $O(N + 1, 2)$. They deformed an $\mathfrak{sl}_2$ triple studied in [4] via the parameter a such that the a-deformed Dunkl harmonic oscillator $\triangle_{k,a} := \|x\|^{2-a} \triangle_k - \|x\|^a$ is symmetric on the Hilbert space $L^2\left(\mathbb{R}^N, \vartheta_{k,a}(x)\, dx\right)$, where $\vartheta_{k,a}(x) = \|x\|^{a-2} h_k(x)$. In the case of $k \equiv 0$, such a-deformed harmonic oscillator is also a deformation of the operator $\|x\| \triangle - \|x\|$ studied by Kobayashi and Mano in [26, 27]. In [25], R. Howe studied a holomorphic semigroup on $L^2(\mathbb{R}^N)$ with the classical harmonic oscillator $\mathbf{H} =: (\triangle - \|x\|^2)/2$ being the infinitesimal generator. The definition of the classical Fourier transform F on $L^2(\mathbb{R}^N)$ was then defined as the boundary value $z = \frac{\pi i}{2}$ of the holomorphic semigroup

$$F := e^{i\pi N/4} \exp\left(\frac{\pi \mathrm{i}}{2}\mathbf{H}\right).$$

Motivated by this definition of the classical Fourier transform on $L^2\left(\mathbb{R}^N\right)$ by Howe, the authors in [8] then proved the existence of a (k,a)-generalized holomorphic semigroup $\mathcal{I}_{k,a}(z)$, $\Re z \geq 0$ with infinitesimal generator $\frac{1}{a}\triangle_{k,a}$ acting on $L^2\left(\mathbb{R}^N, \vartheta_{k,a}(x)\,dx\right)$. The (k,a)-generalized Laguerre semigroup $\mathcal{I}_{k,a}(z) = \exp\left(\frac{z}{a}\triangle_{k,a}\right)$ generalizes the Hermite semigroup studied by Howe [25] ($k \equiv 0$ and $a = 2$), the Laguerre semigroup studied by Kobayashi and Mano [26, 27] ($k \equiv 0$ and $a = 1$), and the Dunkl Hermite semigroup studied by Rösler [35] ($k \geq 0$, $a = 2$ and $z = 2t,\ t > 0$). When taking the boundary value $z = \frac{\pi i}{2}$, the semigroup $\mathcal{I}_{k,a}(z)$ recedes to the so-called (k,a)-generalized Fourier transform $F_{k,a}$, i.e.,

$$F_{k,a} = c\mathcal{I}_{k,a}\left(\frac{\pi i}{2}\right), \tag{1.1}$$

where $c = e^{i\pi\left(\frac{2\langle k\rangle + N + a - 2}{2a}\right)}$ and $\langle k\rangle := \sum_{\alpha\in R^+} k(\alpha)$. The generalized Fourier transform includes the Fourier transform ($k \equiv 0$ and $a = 2$), the Kobayashi-Mano Hankel transform ($k \equiv 0$ and $a = 1$), and the Dunkl transform [20] ($k \geq 0$ and $a = 2$).

We will define a one-dimensional a-deformed Laguerre holomorphic semigroup $I_{a,\alpha;z} := e^{-\frac{z}{a}L_{a,\alpha}}$ with the infinitesimal generator $-\frac{1}{a}L_{a,\alpha}$, where $L_{a,\alpha}$ is the a-deformed Laguerre operator. Then we will give a spherical harmonic expansion of the (k,a)-generalized Laguerre semigroup $\mathcal{I}_{k,a}(z)$ via the a-deformed Laguerre holomorphic semigroups $I_{a,\alpha;z}$, and show that the expansion reduces to the Bochner-type identity when taking the boundary value $z = \frac{\pi i}{2}$. And then we will prove a Hardy inequality for fractional powers of the a-deformed Dunkl harmonic operator $\triangle_{k,a} = \|x\|^{2-a}\triangle_k - \|x\|^a$ using this expansion. When $a = 2$, the fractional Hardy inequality reduces to that of Dunkl-Hermite operators given by Ó. Ciaurri, L. Roncal and S. Thangavelu [12]. The operators $L_{a,\alpha}$ also give a tangible characterization of the radial part of the (k,a)-generalized Laguerre semigroup on each k-spherical component $\mathcal{H}_k^m\left(\mathbb{R}^N\right)$ for $\lambda_{k,a,m} := \frac{2m+2\langle k\rangle+N-2}{a} \geq -1/2$ defined via decomposition of unitary representation.

For the two particular cases when $a = 1$ and $a = 2$ (the Dunkl case) assuming that $2\langle k\rangle + N + a - 3 \geq 0$ of (k,a)-generalized Fourier analysis, the analytic structure is much richer because it is already known that for the two cases the integral kernel $B_{k,a}(x,y)$ of the (k,a)-

generalized Fourier transform, which takes the place of the exponential function $e^{-i\langle x,y\rangle}$ in classical Fourier transform, is uniformly bounded by 1. In this case one can define the (k,a)-generalized translation operator via an integral combining the inversion formula of the (k,a)-generalized Fourier transform for $a=\frac{2}{n},\ n\in\mathbb{N}$. And for the two cases we have the formula of the generalized translation operator for radial functions. For the case of $a=2$ (Dunkl theory), the radial formula was found by Rösler [36] and the analytic structure of Dunkl theory was intensively studied in the past thirty years, including the study of maximal functions, Bochner–Riesz means, multipliers, Riesz transforms and Calderón–Zygmund theory. For the case of $a=1$, S. Ben Saïd and L. Deleaval derived the radial formula of the generalized translation in [5] and also found many parallel results to Dunkl's analysis. The study for this case is still at its infancy. We will study the generalized translation in the two cases and characterize the support of the translation on radial functions.

We will also define and investigate the imaginary powers $(-\triangle_{k,a})^{-i\sigma}, \sigma\in\mathbb{R}$ of the (k,a)-generalized harmonic oscillator $-\triangle_{k,a}$ for the two cases when $a=2$ and 1 respectively, and prove the L^p-boundedness $(1<p<\infty)$ and weak L^1-boundedness of such operators. To prove this result, we develop the Calderón–Zygmund theory adapted to the (k,a)-generalized setting ($a=2$ and 1). For the case when $a=2$ (Dunkl setting), the adapted Calderón–Zygmund theory was already developed by Amri and Sifi [2] to prove the L^p-boundedness $(1<p<\infty)$ and weak L^1-boundedness of the Dunkl Riesz transform. For the case when $a=1$, we need to construct the metric space of homogeneous type corresponding to the $(k,1)$-generalized setting first according to the radial formula for the $(k,1)$-generalized translation, in order to adapt the Calderón–Zygmund theory on general homogeneous spaces to the $(k,1)$-generalized setting. And it will be shown that the imaginary powers $(-\triangle_{k,1})^{-i\sigma}$ are singular integral operators satisfying the corresponding Hörmander type condition, which motivates us to develop the Calderón–Zygmund theory in $(k,1)$-generalized setting.

The material is divided into six chapters. In Chapter 2 we review the motivation and some main results in Dunkl theory and the (k,a)-generalized Fourier analysis developed by S. Ben Saïd, T. Kobayashi and B. Ørsted. For these results we refer to [7, 8, 18, 20]. In Chapter 3, we give the definitions of the a-deformed Laguerre operators and holomorphic semigroup, and then prove the fractional Hardy inequality for the (k,a)-generalized harmonic operator $-\triangle_{k,a}$ using the spherical harmonic expansion of the (k,a)-generalized Laguerre semigroup.

We will also show the relationship between the expansion with $\mathfrak{sl}_2$-representation in this chapter. The results in this chapter are based on my paper [41]. In Chapter 4, we study the (k, a)-generalized translation operator, and characterize the support of the generalized translation on radial functions for $a = 2$ and 1 respectively from the radial formulas. The study of the support of the generalized translation is based on my paper [39] and [40] for $a = 2$ and 1 respectively. For the particular case when $a = 1$, we will study the metric space corresponding to the $(k, 1)$-generalized setting, which is contained in the paper [40]. In Chapter 5, we develop the Calderón–Zygmund theory adapted to the (k, a)-generalized Fourier analysis for $a = 2$ and 1 (see [2] for the Calderón–Zygmund theory adapted to the case when $a = 2$, the Dunkl case and [40] for the theory adapted to the case when $a = 1$). In Chapter 6, we define and investigate the imaginary powers $(-\triangle_{k,a})^{-i\sigma}$ for $a = 2$ and 1 and show that they satisfy the corresponding Hörmander type condition given in Chapter 5 to prove the L^p-boundedness $(1 < p < \infty)$ and weak L^1-boundedness. The results for the case when $a = 1$ in Chapter 5 and 6 are under the condition that $k > 0$ and $2\langle k\rangle + N - 2 > 0$ and are contained in my paper [40].

2 Preliminaries

2.1 Classical spherical harmonics

In this subsection we review the classical theory of spherical harmonics and some properties. For $x \in \mathbb{R}^N$, we write $x = (x_1, \ldots, x_N)$. For any $x, y \in \mathbb{R}^N$, denote by $\langle x, y\rangle = \sum_{j=1}^N x_j y_j$ the standard inner product associated with norm $\|x\|$. For $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}_0^N$, denote $x^\alpha := x_1^{\alpha_1} \cdots x_N^{\alpha_N}$ and it has degree $|\alpha| = \alpha_1 + \cdots + \alpha_N$. A homogeneous polynomial p of degree m is defined as $p(x) = \sum_{|\alpha|=m} c_\alpha x^\alpha$, where c_α are real or complex numbers. Let P_m be the space of real homogeneous polynomials on $\mathbb{R}^N$ of degree m. And let ∂_i be the partial derivative and $\triangle = \sum_{i=1}^N \partial_i^2$ be the Euclidean Laplacian. The homogeneous polynomials satisfying $\triangle p = 0$ are called harmonic polynomials. Let $\mathcal{H}^m(\mathbb{R}^N)$ be the space of harmonic polynomials of degree m. The restrictions of the elements of the spaces $\mathcal{H}^m(\mathbb{R}^N)$ on the unit sphere $\mathbb{S}^{N-1}$, to be denoted as $\mathcal{H}^m(\mathbb{R}^N)|_{\mathbb{S}^{N-1}}$, are called spherical harmonics. The spherical harmonics $\mathcal{H}^m(\mathbb{R}^N)|_{\mathbb{S}^{N-1}},\ m = 0, 1, 2, \ldots$ are orthogonal to each other with respect to the inner product

$$\langle f, g\rangle_{\mathbb{S}^{N-1}} := \frac{\Gamma(N/2)}{2\pi^{N/2}} \int_{\mathbb{S}^{N-1}} f(x)g(x)d\sigma(x),$$

where $d\sigma$ is the surface measure. This can be shown in the following theorem.

Theorem 2.1. *(See, e.g., [16, Theorem 1.1.2]) For $Z_n \in \mathcal{H}^n(\mathbb{R}^N)$, $Z_m \in \mathcal{H}^m(\mathbb{R}^N)$, and $n \neq m$, we have $\langle Z_n, Z_m\rangle_{\mathbb{S}^{N-1}} = 0$.*

Proof. For $x = rx'(r > 0,\ x' \in \mathbb{S}^{N-1})$, we have $Z_m(x) = r^m Z_m(x')$ since Z_m is homogeneous. So $\frac{\partial Z_m}{\partial r}(x') = mZ_m(x')$. By Green's identity,

$$(n-m)\int_{\mathbb{S}^{N-1}} Z_n Z_m d\sigma = \int_{\mathbb{S}^{N-1}} \left(Z_n \frac{\partial Z_m}{\partial r} - Z_m \frac{\partial Z_n}{\partial r}\right) d\sigma = \int_{\mathbb{B}^N} \left(Z_n \triangle Z_m - Z_m \triangle Z_n\right) dx = 0,$$

where $\mathbb{B}^N$ is the unit ball in $\mathbb{R}^N$, since $\triangle Z_m = \triangle Z_n = 0$. □

The Laplace-Beltrami operator $\triangle_0$ is the restriction of the Laplace operator $\triangle$ to the unit sphere $\mathbb{S}^{N-1}$. It satisfies

$$\triangle = \frac{\partial^2}{\partial r^2} + \frac{N-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\triangle_0,$$

where

$$\triangle_0 = \sum_{i=1}^{N-1} \frac{\partial^2}{\partial \xi_i^2} - \sum_{i=1}^{N-1}\sum_{j=1}^{N-1} \xi_i \xi_j \frac{\partial^2}{\partial \xi_i \partial \xi_j} - (N-1)\sum_{i=1}^{N-1} \xi_i \frac{\partial}{\partial \xi_i}$$

for the polar coordinates $x = r\xi,\ r > 0,\ \xi \in \mathbb{S}^{N-1}$. And the spherical harmonics are eigenfunctions of the Laplace-Beltrami operator $\triangle_0$, i.e.,

$$\triangle_0 Z(\xi) = -m(m+N-2)Z(\xi),\ Z \in \mathcal{H}^m(\mathbb{R}^N),\ \xi \in \mathbb{S}^{N-1}.$$

The following is the spherical harmonic decomposition for $L^2\left(\mathbb{S}^{N-1}, d\sigma(x')\right)$,

$$L^2\left(\mathbb{S}^{N-1}, d\sigma(x')\right) = \sum_{m\in\mathbb{N}}^{\oplus} \mathcal{H}^m\left(\mathbb{R}^N\right)|_{\mathbb{S}^{N-1}},$$

where $d\sigma$ is the spherical measure on $\mathbb{S}^{N-1}$. For each fixed spherical harmonic $\mathcal{H}^m\left(\mathbb{R}^N\right)|_{\mathbb{S}^{N-1}}$, we take an orthonormal basis of the space as

$$\left\{Z_i^m :\ i = 1, 2, \cdots, n(m)\right\},$$

where $n(m) = \dim\left(\mathcal{H}^m\left(\mathbb{R}^N\right)|_{\mathbb{S}^{N-1}}\right)$. Then we have the spherical harmonic expansion of $f \in L^2\left(\mathbb{R}^N\right)$,

$$f\left(rx'\right) = \sum_{m=0}^{\infty}\sum_{i=1}^{n(m)} f_{m,i}(r) Z_i^m(x'), r > 0,\ x' \in \mathbb{S}^{N-1},$$

where

$$f_{m,i}(r) = \int_{\mathbb{S}^{N-1}} f\left(rx'\right) Z_i^m(x') d\sigma(x').$$

2.2 The G-invariant measure

For any nonzero vector $\alpha \in \mathbb{R}^N$, define the reflection σ_α with respect to the hyperplane $\alpha^\perp$ orthogonal to α,

$$\sigma_\alpha(x) = x - 2\frac{\langle x, \alpha\rangle}{\|\alpha\|^2}\alpha.$$

A finite set $R \subset \mathbb{R}^N \backslash \{0\}$ is called a (*reduced*) *root system* if it satisfies the following conditions:

i. $R \cap \mathbb{R}\alpha = \{\pm\alpha\}$, for any $\alpha \in R$;

ii. $\sigma_\alpha(R) = R$, for any $\alpha \in R$;

In addition, a root system is called a crystallographic root system if it satisfies the condition $2\frac{\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}$ for any $\alpha, \beta \in R$. We consider reduced but not necessarily crystallographic root systems. Given a root system R, the finite subgroup G of $O(N)$ generated by the reflections σ_α is called the *Coxeter group* (*relection group*) of the root system. Define a *multiplicity* *function* $k : R \to \mathbb{C}$ such that k is G-invariant, that is, $k(\alpha) = k(\beta)$ if σ_α and σ_β are conjugate. We assume that $k \geqslant 0$.

In the last 80's, C.F. Dunkl [20] gave a far-reaching generalization of Euclidean Fourier analysis related to root system with a rich structure parallel to ordinary Fourier analysis, where the finite reflection groups play the role of orthogonal groups in Euclidean Fourier analysis. The Lebesgue measure was replaced by a weighted measure $dm_k(x) = h_k(x)dx$, where

$$h_k(x) = \prod_{\alpha\in R} |\langle \alpha,\ x\rangle|^{k(\alpha)}.$$

It is observed that

$$m_k(B(x,r)) \sim r^N \prod_{\alpha\in R} \left(|\langle x,\alpha\rangle| + r\right)^{k(\alpha)},$$

where $B(x,r) = \{y \in \mathbb{R}^N : \|x-y\| \leq r\}$, and so m_k is a doubling measure, that is, there is a constant $C > 0$ such that

$$m_k(B(x,2r)) \leq Cm_k(B(x,r))$$

for $x \in \mathbb{R}^N,\ r > 0$.

The measure parameterized by the multiplicity function k is invariant under the reflection group G. It is in substitute of the Lebesgue measure dx in classical analysis, which is invariant under the orthogonal group $O(N)$. So, in this sense we say that the finite reflection group G plays the role of the orthogonal group in classical analysis. Such generalization of classical analysis, called Dunkl theory, has been intensively studied in the past thirty years. We denote $\mathbf{N} = N + \sum_{\alpha\in R} k(\alpha)$ to be the homogeneous dimension of the root system.

2.3 Spherical h-harmonics

The study of Dunkl theory originates from a generalization of spherical harmonics, called spherical h-harmonics, where the Dunkl weight measure $dm_k(x) = h_k(x)dx$ plays the role of Lebesgue measure dx in the classical theory of spherical harmonics. Dunkl [18] constructed an operator $\triangle_k$ analogous to the classical Laplacian such that the intersection of the kernel of the operator with the space of homogeneous polynomials of degree m are orthogonal to each other with respect to the Dunkl weight measure $dm_k(x)$, for $m = 0, 1, 2, \cdots$. The operator $\triangle_k$ is called Dunkl Laplacian and has the following explicit expression for normalized root systems (i.e., $\|\alpha\| = \sqrt{2}$),

$$\triangle_k = D_k - E_k,$$

with

$$D_k f(x) = \triangle f(x) + 2 \sum_{\alpha \in R^+} k(\alpha) \frac{\langle \nabla f, \alpha \rangle}{\langle \alpha, x \rangle},$$

where ∇ is the Euclidean gradient and R^+ is any fixed positive subsystem of R, and

$$E_k f(x) = 2 \sum_{\alpha \in R^+} k(\alpha) \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle^2}.$$

The minus Dunkl Laplacian $-\triangle_k$ is essentially self-adjoint on $L^2(\mathbb{R}^N, h_k(x)dx)$ and positive definite, and so $\triangle_k$ is the generator of the contraction semigroup $\left\{e^{t\triangle_k}\right\}_{t\geq 0}$. The elements in the space $\mathcal{H}_k^m(\mathbb{R}^N) := P_m \cap ker\triangle_k$ are h-harmonic polynomials of degree m. And the restrictions $\mathcal{H}_k^m(\mathbb{R}^N)|_{\mathbb{S}^{N-1}}$ of the spaces $\mathcal{H}_k^m(\mathbb{R}^N)$ to the unit sphere $\mathbb{S}^{N-1}$ are called spherical h-harmonics. The following theorem shows that the spherical h-harmonics of different degree are orthogonal to each other with respect to the inner product

$$\langle f, g \rangle_{\mathbb{S}^{N-1};k} := \frac{1}{\omega_k} \int_{\mathbb{S}^{N-1}} f(x)g(x)h_k(x)d\sigma(x),$$

where $\omega_k = \int_{\mathbb{S}^{N-1}} h_k(x)d\sigma(x)$.

Theorem 2.2. *([17]) If $f \in \mathcal{H}_k^n(\mathbb{R}^N)$, $g \in \mathcal{H}_k^m(\mathbb{R}^N)$ and $n \neq m$, then $\langle f, g \rangle_{\mathbb{S}^{N-1};k} = 0$.*

Proof. We claim the following analogue of the classical Green's identity without proof first,

$$\int_{\mathbb{S}^{N-1}} \frac{\partial f}{\partial n} g h_k d\sigma = \int_{\mathbb{B}^N} \left(gD_k f + \langle \nabla f, \nabla g \rangle\right) h_k dx,$$

where $\frac{\partial f}{\partial n}$ denotes the normal derivative of f. Since $\frac{\partial f}{\partial n} = nf$ because f is homogeneous of degree n, and $\triangle_k f = \triangle_k g = 0$, we have

$$(n-m) \int_{\mathbb{S}^{N-1}} fgh_k d\sigma = \int_{\mathbb{B}^N} \left(gD_k f - fD_k g\right) h_k dx = \int_{\mathbb{B}^N} \left(gE_k f - fE_k g\right) h_k dx = 0.$$

The last equality is from the symmetry of E_k with respect to $\langle \cdot, \cdot \rangle_{\mathbb{S}^{N-1};k}$ (using polar coordinates). □

The spaces $\mathcal{H}_k^m(\mathbb{R}^N)|_{\mathbb{S}^{N-1}}$, $m = 0, 1, \cdots$ are finite dimensional and there is the spherical harmonics decomposition

$$L^2\left(\mathbb{S}^{N-1}, h_k(x') d\sigma(x')\right) = \sum_{m\in\mathbb{N}}^{\oplus} \mathcal{H}_k^m(\mathbb{R}^N)|_{\mathbb{S}^{N-1}}. \tag{2.1}$$

For each fixed $m \in \mathbb{N}$, denote by $d(m) = \dim \left(\mathcal{H}_k^m \left(\mathbb{R}^N\right)|_{\mathbb{S}^{N-1}}\right)$. Let

$$\{Y_i^m :\ i = 1, 2, \cdots, d(m)\} \tag{2.2}$$

be an orthonormal basis of $\mathcal{H}_k^m \left(\mathbb{R}^N\right)|_{\mathbb{S}^{N-1}}$. They are the eigenvectors of the generalized Laplace–Beltrami operator $\triangle_k|_{\mathbb{S}^{N-1}}$, i.e.,

$$\triangle_k|_{\mathbb{S}^{N-1}} Y_i^m(\xi) = -m(m + \mathbf{N} - 2)Y_i^m(\xi),\ \xi \in \mathbb{S}^{N-1}.$$

2.4 The intertwining operator

Let $\{e_i,\ i = 1, 2, ..., N\}$ be the canonical orthogonal basis in $\mathbb{R}^N$. The *Dunkl operators* $\{T_i :\ 1 \le i \le N\}$ introduced in [18] were constructed such that $\triangle_k = \sum_{i=1}^N T_i^2$. It is the deformations by difference operators of directional derivatives and can be expressed explicitly as follows for normalized root systems:

$$T_i f(x) = \partial_i f(x) + \sum_{\alpha \in R} \frac{k(\alpha)}{2} \langle \alpha,\ e_i \rangle \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha,\ x \rangle}.$$

They commute pairwise and are skew-symmetric with respect to the G-invariant measure $dm_k(x) = h_k(x)dx$.

The operators ∂_i and T_i are intertwined by the Laplace-type operator (see [19])

$$V_k f(x) = \int_{\mathbb{R}^N} f(y) d\mu_x(y),$$

associated to a family of probability measures $\left\{\mu_x |\ x \in \mathbb{R}^N\right\}$ with compact support, that is,

$$T_i \circ V_k = V_k \circ \partial_i.$$

Specifically, the support of μ_x is contained in the convex hull $co(G.x)$, where $G.x = \{gx |\ g \in G\}$ is the orbit of x. For any Borel set B and any $r > 0$, $g \in G$, the probability measures satisfy

$$\mu_{rx}(B) = \mu_x\left(r^{-1}B\right),\ \mu_{gx}(B) = \mu_x\left(g^{-1}B\right).$$

The intertwining operator V_k is one of the most important operators in Dunkl theory.

The joint eigenfunction $E(x, y)$ of the Dunkl operators $\{T_i :\ 1 \le i \le N\}$ (or the eigenfunction of the Dunkl Laplacian $\triangle_k$) for fixed y is the integral kernel of the generalized

Fourier transform F_k, called Dunkl transform (see [20]). It takes the place of the exponential function $e^{\langle x,y\rangle}$ in classical Fourier transform, i.e.,

$$F_k(f)(\xi) := \frac{1}{c_k}\int_{\mathbb{R}^N} f(x)E(-i\xi,\ x)dm_k(x),\ c_k = \int_{\mathbb{R}^N} e^{-\frac{\|x\|^2}{2}}dm_k(x),\ f \in L^1(m_k).$$

The eigenfunction $E(x,y)$ is called Dunkl kernel and can be expressed via the intertwining operator as

$$E(x,y) = V_k\left(e^{\langle\cdot,y\rangle}\right)(x) = \int_{\mathbb{R}^N} e^{\langle\eta,y\rangle}d\mu_x(\eta).$$

2.5 The (k,a)-generalized harmonic oscillator

Take a basis of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ as

$$\mathbf{e}^+ := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\quad \mathbf{e}^- := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\quad \mathbf{h} := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The triple $\{\mathbf{e}^+,\mathbf{e}^-,\mathbf{h}\}$ is an $\mathfrak{sl}_2$ triple, i.e.,

$$\left[\mathbf{e}^+,\mathbf{e}^-\right] = \mathbf{h},\ \left[\mathbf{h},\mathbf{e}^+\right] = 2\mathbf{e}^+,\ \left[\mathbf{h},\mathbf{e}^-\right] = -2\mathbf{e}^-.$$

In [8], the authors gave a deformation of an $\mathfrak{sl}_2$ triple studied in [4] via a parameter a

$$\mathbb{E}^+_{k,a} := \frac{i}{a}\|x\|^a,\ \mathbb{E}^-_{k,a} := \frac{i}{a}\|x\|^{2-a}\Delta_k,\ \mathbb{H}_{k,a} := \frac{2}{a}\sum_{i=1}^N x_i\partial_i + \frac{N+2\langle k\rangle+a-2}{a}.$$

These differential-difference operators also form an $\mathfrak{sl}_2$ triple. With these operators, the Dunkl harmonic oscillator $\triangle_k - \|x\|^2$ is deformed to be the (k,a)-generalized harmonic oscillator as

$$\Delta_{k,a} := ia\left(\mathbb{E}^+_{k,a} - \mathbb{E}^-_{k,a}\right) = \|x\|^{2-a}\triangle_k - \|x\|^a.$$

It is symmetric on the Hilbert space $L^2\left(\mathbb{R}^N,\vartheta_{k,a}(x)\,dx\right)$, where $\vartheta_{k,a}(x) = \|x\|^{a-2}h_k(x)$. The case when $k \equiv 0,\ a = 1$ (the operator $\|x\|\triangle - \|x\|$) was studied by T. Kobayashi and G. Mano in [26, 27]. When $a = 2$, the (k,a)-generalized harmonic oscillator $\Delta_{k,a}$ recedes to the Dunkl harmonic oscillator.

Define the representation $\omega_{k,a}$ of $\mathfrak{sl}(2,\mathbb{R})$ on $C^\infty\left(\mathbb{R}^N\setminus\{0\}\right)$

$$\omega_{k,a} : \mathfrak{sl}(2,\mathbb{R}) \to \mathrm{End}\left(C^\infty\left(\mathbb{R}^N\setminus\{0\}\right)\right)$$

by setting

$$\omega_{k,a}(\mathbf{h}) = \mathbb{H}_{k,a},\ \ \omega_{k,a}\left(\mathbf{e}^+\right) = \mathbb{E}^+_{k,a},\ \ \omega_{k,a}\left(\mathbf{e}^-\right) = \mathbb{E}^-_{k,a}.$$

Denote by $U(\mathfrak{sl}(2,\mathbb{C}))$ the universal enveloping algebra of the complex Lie algebra $\mathfrak{sl}(2,\mathbb{C}) \simeq \mathfrak{sl}(2,\mathbb{R}) \otimes \mathbb{C}$. Then we can extend the representation to a $\mathbb{C}$-algebra homomorphism

$$\omega_{k,a} : U(\mathfrak{sl}(2,\mathbb{C})) \to \mathrm{End}\left(C^\infty\left(\mathbb{R}^N \setminus \{0\}\right)\right).$$

Set

$$\mathbf{k} := \frac{1}{i}\left(\mathbf{e}^+ - \mathbf{e}^-\right),\ \mathbf{n}^+ := \tfrac{1}{2}(i\mathbf{h} - \mathbf{e}^+ - \mathbf{e}^-),\ \mathbf{n}^- := -\tfrac{1}{2}(i\mathbf{h}+\mathbf{e}^+ + \mathbf{e}^-).$$

Then we can interpret $\Delta_{k,a}$ as $\mathfrak{sl}_2$ representation

$$\omega_{k,a}(\mathbf{k}) = \frac{1}{i}\left(\mathbb{E}^+_{k,a} - \mathbb{E}^-_{k,a}\right) = \frac{\|x\|^a - \|x\|^{2-a}\Delta_k}{a} = -\frac{1}{a}\Delta_{k,a}.$$

2.6 An orthonormal basis in $L^2\left(\mathbb{R}^N, \vartheta_{k,a}(x)\,dx\right)$

Consider the weight function $\vartheta_{k,a}(x) = \|x\|^{a-2} h_k(x)$. It reduces to $h_k(x)$ when $a = 2$ and for any $x' \in \mathbb{S}^{N-1}$,

$$\vartheta_{k,a}(x') = h_k(x').$$

For the polar coordinates $x = rx'(r > 0,\ x' \in \mathbb{S}^{N-1})$,

$$\vartheta_{k,a}(x)\,dx = r^{2\langle k\rangle + N + a - 3}\vartheta_{k,a}(x')\,dr d\sigma(x').$$

From the spherical harmonic decomposition (2.1) of $L^2\left(\mathbb{S}^{N-1}, h_k(x')\,d\sigma(x')\right)$, there is a unitary isomorphism (see [8, (3.25)])

$$\sum_{m\in\mathbb{N}}^{\oplus}\left(\mathcal{H}^m_k\left(\mathbb{R}^N\right)|_{\mathbb{S}^{N-1}}\right) \otimes L^2\left(\mathbb{R}_+, r^{\,2\langle k\rangle+N+a-3}dr\right) \xrightarrow{\sim} L^2\left(\mathbb{R}^N, \vartheta_{k,a}(x)\,dx\right).$$

Define the Laguerre polynomial as

$$L^\mu_l(t) := \sum_{j=0}^{l} \frac{(-1)^j \Gamma(\mu + l + 1)}{(l-j)!\Gamma(\mu + j + 1)} \frac{t^j}{j!},\ \mathrm{Re}\mu > -1.$$

Proposition 2.3. *([8, Proposition 3.15]) For fixed $m \in \mathbb{N}$, $a > 0$, and a multiplicity function k satisfying $\lambda_{k,a,m} := \frac{2m+2\langle k\rangle+N-2}{a} > -1$. Set*

$$\psi^{(a)}_{l,m}(r) := \left(\frac{2^{\lambda_{k,a,m}+1}\Gamma(l+1)}{a^{\lambda_{k,a,m}}\Gamma(\lambda_{k,a,m}+l+1)}\right)^{1/2} r^m L^{\lambda_{k,a,m}}_l\left(\frac{2}{a}r^a\right)\exp\left(-\frac{1}{a}r^a\right). \tag{2.3}$$

Then $\left\{\psi^{(a)}_{l,m}(r) :\ l \in \mathbb{N}\right\}$ forms an orthonormal basis in $L^2\left(\mathbb{R}_+, r^{\,2\langle k\rangle+N+a-3}dr\right)$.

Combining the orthonormal basis (2.2) of $\mathcal{H}_k^m\left(\mathbb{R}^N\right)|_{\mathbb{S}^{N-1}}$, Proposition 2.3 yields the orthonormal basis in $L^2\left(\mathbb{R}^N,\vartheta_{k,a}(x)\,dx\right)$ immediately.

Corollary 2.4. *([8, Corollary 3.17]) Suppose $a>0$ and k satisfy that $2m+2\langle k\rangle+N+a-2>0$, Set*

$$\Phi_{l,m,j}^{(a)}(x) := Y_j^m\left(\frac{x}{\|x\|}\right)\psi_{l,m}^{(a)}(\|x\|). \tag{2.4}$$

Then

$$\left\{\Phi_{l,m,j}^{(a)}\,\middle|\, l\in\mathbb{N},\ m\in\mathbb{N},\ j=1,2,\cdots,d(m)\right\} \tag{2.5}$$

forms an orthonormal basis of $L^2\left(\mathbb{R}^N,\vartheta_{k,a}(x)\,dx\right)$.

Denote by

$$W_{k,a}\left(\mathbb{R}^N\right) := \mathbb{C}\text{-}span\left\{\Phi_l^{(a)}(p,\cdot)\,\middle|\, l\in\mathbb{N},\ m\in\mathbb{N},\ p\in\mathcal{H}_k^m\left(\mathbb{R}^N\right)\right\},$$

where

$$\Phi_l^{(a)}(p,x) = p(x')r^m L_l^{\lambda_{k,a,m}}\left(\frac{2}{a}r^a\right)\exp\left(-\frac{1}{a}r^a\right)$$

for $x=rx'$ $(r>0,\ x'\in\mathbb{S}^{N-1})$. It is a dense subset of the Hilbert space $L^2\left(\mathbb{R}^N,\vartheta_{k,a}(x)\,dx\right)$. It was shown in [8, Theorem 3.19] that $\Phi_l^{(a)}(p,\cdot)$ are eigenfunctions for $-\Delta_{k,a}$, i.e.,

$$\omega_{k,a}(\mathbf{k})\Phi_l^{(a)}(p,x) = \left(2l+\lambda_{k,a,m}+1\right)\Phi_l^{(a)}(p,x). \tag{2.6}$$

Let $\mathfrak{g}=\mathfrak{sl}(2,\mathbb{R})$ and K be a maximal compact modulo center subgroup of the universal covering group $\widetilde{SL(2,\mathbb{R})}$ of $SL(2,\mathbb{R})$. Then $\left(\omega_{k,a},W_{k,a}\left(\mathbb{R}^N\right)\right)$ is a $G\times(\mathfrak{g}_{\mathbb{C}},K)$-module and can be decomposed as (see [8, Theorem 3.28])

$$W_{k,a}\left(\mathbb{R}^N\right) \simeq \bigoplus_{m=0}^{\infty}\mathcal{H}_k^m(\mathbb{R}^N)|_{S^{N-1}}\otimes\pi_K(\lambda_{k,a,m}), \tag{2.7}$$

where $\pi_K(\lambda_{k,a,m})$ is $\mathfrak{sl}(2,\mathbb{R})$ acting on the vector space $\mathbb{C}$-span$\{\Phi_l^{(a)}(p,\cdot) : l\in\mathbb{N}\}$ for fixed $p\in\mathcal{H}_k^m(\mathbb{R}^N)$. It is the irreducible lowest weight module of weight $\lambda_{k,a,m}+1$.

2.7 The (k,a)-generalized Laguerre semigroup and Fourier transform

It is observed the fact that the (k,a)-generalized harmonic oscillator $\triangle_{k,a}$ is an essentially self-adjoint operator on $L^2\left(\mathbb{R}^N,\vartheta_{k,a}(x)\,dx\right)$ with only negative discrete spectrum. And so it is the infinitesimal generator of the corresponding contraction semigroup. Thus, for

$a + 2\langle k\rangle + N - 2 > 0$, the infinitesimal representation $\omega_{k,a}$ of $\mathfrak{sl}(2,\mathbb{R})$ can be lifted to a unique unitary representation $\Omega_{k,a}$ of the universal covering group $\widetilde{SL(2,\mathbb{R})}$ of $SL(2,\mathbb{R})$ on the Hilbert space $L^2\left(\mathbb{R}^N, \vartheta_{k,a}(x)\,dx\right)$ (see [8, Theorem 3.30]), i.e.,

$$\omega_{k,a}(X) = \left.\frac{d}{dt}\right|_{t=0} \Omega_{k,a}\left(Exp(tX)\right),\ X \in \mathfrak{sl}(2,\mathbb{R}) \tag{2.8}$$

on the dense subset $W_{k,a}\left(\mathbb{R}^N\right)$ of $L^2\left(\mathbb{R}^N, \vartheta_{k,a}(x)\,dx\right)$. We can then define the (k,a)-generalized Laguerre holomorphic semigroup on $L^2\left(\mathbb{R}^N, \vartheta_{k,a}(x)\,dx\right)$ as ((see [8]))

$$\mathcal{I}_{k,a}(z) := \Omega_{k,a}(\gamma_z) = \exp\left(\frac{z}{a}\triangle_{k,a}\right),\ \gamma_z = \mathrm{Exp}\left(-\mathrm{z}\mathbf{k}\right)\ \Re z \geq 0.$$

It has the following spectral decomposition on $L^2\left(\mathbb{R}^N, \vartheta_{k,a}(x)\,dx\right)$ via the basis (2.5)

$$\mathcal{I}_{k,a}(z)f(x) = \sum_{l,m,j} e^{-z\left(2l+\lambda_{k,a,m}+1\right)} \left\langle f, \Phi^{(a)}_{l,m,j}\right\rangle_{k,a} \Phi^{(a)}_{l,m,j}(x), \tag{2.9}$$

where $\langle f,g\rangle_{k,a} = \int_{\mathbb{R}^N} f(x)g(x)\vartheta_{k,a}(x)dx$. By Schwartz kernel theorem, it has an integral representation on $L^2\left(\mathbb{R}^N, \vartheta_{k,a}(x)\,dx\right)$ by means of a distribution kernel $\Lambda_{k,a}(x,y;z)$ (see [8, (4.56)])

$$\mathcal{I}_{k,a}(z)f(x) = c_{k,a}\int_{\mathbb{R}^N} f(y)\,\Lambda_{k,a}(x,y;z)\,\vartheta_{k,a}(y)\,dy, \tag{2.10}$$

where $c_{k,a} = \left(\int_{\mathbb{R}^N}\exp\left(-\frac{1}{a}\|x\|^a\right)\vartheta_{k,a}(x)\,dx\right)^{-1}$ and

$$\Lambda_{k,a}(r\omega, s\eta; z) := \left(\widetilde{V}_k h_{k,a}(r,s;z;\cdot)\right)(\omega,\eta) \tag{2.11}$$

for $x = r\omega$, $y = s\eta$, $r,s > 0$ and $\omega,\eta \in \mathbb{S}^{N-1}$. Here $\widetilde{V}_k$ is defined by $\left(\widetilde{V}_k h\right)(x,y) := \left(V_k h_y\right)(x)$, where $h_y(\cdot) := h\left(\langle\cdot,y\rangle\right)$ for a continuous function $h(t)$ of one variable. And $h_{k,a}(r,s;z;w)$ has its closed formula

$$h_{k,a}(r,s;z;t) = \frac{\exp(-\frac{1}{a}(r^a+s^a)\coth(z))}{\sinh(z)^{\frac{2\langle k\rangle+N+a-2}{a}}} \times \begin{cases} \Gamma\Big(\langle k\rangle + \dfrac{N-1}{2}\Big)\widetilde{I}_{\langle k\rangle+\frac{N-3}{2}}\Big(\dfrac{\sqrt{2}(rs)^{\frac{1}{2}}}{\sinh z}(1+t)^{\frac{1}{2}}\Big) & (a=1), \\ \exp\Big(\dfrac{rst}{\sinh z}\Big) & (a=2), \end{cases} \tag{2.12}$$

where $\widetilde{I}_v$ is the normalized I-Bessel function and has the following integral formula (see, e.g., [44, 6.15 (2)])

$$\widetilde{I}_v(w) = \frac{1}{\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)}\int_{-1}^{1} e^{wu}\left(1-u^2\right)^{v-\frac{1}{2}}du,\ \nu > -1/2,\ w\in\mathbb{C}.$$

The integral on the right hand side of (2.10) converges absolutely for all $f \in L^2\left(\mathbb{R}^N, \vartheta_{k,a}(x)\,dx\right)$ if $\Re z > 0$ and for all $f \in (L^1 \cap L^2)\left(\mathbb{R}^N, \vartheta_{k,a}(x)\,dx\right)$ if $\Re z = 0$ (see [8, Corollary 4.28]). From (2.11) and (2.12) we get an expression of $\Lambda_{k,a}(x,y;z)$ (a slight modification of Proposition 5.10 in [8])

$$\Lambda_{k,a}(x,y;z) = \frac{\exp(-\frac{1}{a}(\|x\|^a + \|y\|^a)\coth(z))}{\sinh(z)^{\frac{2\langle k\rangle+N+a-2}{a}}} \times \begin{cases} \Gamma\Big(\langle k\rangle + \dfrac{N-1}{2}\Big) V_k\left(\widetilde{I}_{\langle k\rangle+\frac{N-3}{2}}\Big(\dfrac{1}{\sinh z}\sqrt{2(\|x\|\,\|y\| + \langle x,\cdot\rangle)}\Big)\right)(y) & (a=1), \\ V_k\left(e^{\frac{1}{\sinh z}\langle x,\cdot\rangle}\right)(y) & (a=2). \end{cases} \tag{2.13}$$

Let

$$B_{k,a}(x,y) := e^{i\pi(\frac{2\langle k\rangle+N+a-2}{2a})}\Lambda_{k,a}\left(x,y;i\frac{\pi}{2}\right).$$

Then the (k,a)-generalized Fourier transform on $L^2\left(\mathbb{R}^N, \vartheta_{k,a}(x)\,dx\right)$ can be expressed as

$$F_{k,a}f(\xi) = c_{k,a}\int_{\mathbb{R}^N} f(y)\,B_{k,a}(\xi,y)\,\vartheta_{k,a}(y)\,dy, \quad \xi \in \mathbb{R}^N$$

because $F_{k,a} := e^{i\pi(\frac{2\langle k\rangle+N+a-2}{2a})}\mathcal{I}_{k,a}\left(\frac{\pi i}{2}\right)$. For $a + 2\langle k\rangle + N - 2 > 0$, it is a unitary operator on $L^2\left(\mathbb{R}^N, \vartheta_{k,a}(x)\,dx\right)$ (see [8, Theorem 5.1]), that is, $\|F_{k,a}(f)\|_{2,k} = \|f\|_{2,k}$ for any $f \in L^2\left(\mathbb{R}^N, \vartheta_{k,a}(x)\,dx\right)$, where $\|f\|_{2,k} := \left(\int_{\mathbb{R}^N} |f(x)|^2\,\vartheta_{k,a}(x)\,dx\right)^{1/2}$. And the Laguerre function $\Phi_l^{(a)}(p,x)$ is an eigenfunction of $F_{k,a}$, i.e.,

$$F_{k,a}\left(\Phi_l^{(a)}(p,\cdot)\right) = e^{-i\pi(l+\frac{m}{a})}\left(\Phi_l^{(a)}(p,\cdot)\right).$$

As the distribution kernel $B_{k,a}(x,y)$ of the (k,a)-generalized Fourier transform for fixed y is the eigenfunction of the operator $\|x\|^{2-a}\triangle_k$ (see [8, Theorem 5.7]), we can consider $\|x\|^{2-a}\triangle_k$ as the a-deformed Dunkl Laplacian in (k,a)-generalized Fourier analysis. The kernel $B_{k,a}(x,y)$ has the following properties:

1. $B_{k,a}(ax,y) = B_{k,a}(x,ay)$ for $a > 0$;
2. $B_{k,a}(gx,gy) = B_{k,a}(x,y)$ for $g \in G$;
3. $B_{k,a}(x,y) = B_{k,a}(y,x)$;
4. $B_{k,a}(0,y) = 1$.

And we have the inversion formulae of the (k, a)-generalized Fourier transform (see [8, Theorem 5.3]), i.e., $(F_{k,a})^{-1} = F_{k,a}$ if $a = \frac{1}{r}$ and $\left(F_{k,a}^{-1} f\right)(x) = (F_{k,a} f)(-x)$ if $a = \frac{2}{2r-1}$, where $r \in \mathbb{N}_+$.

In [8, Theorem 5.11], the authors showed that the integral kernel $B_{k,a}(x, y)$ satisfies the condition

$$|B_{k,a}(x, y)| \leq B_{k,a}(0, y) = 1 \tag{2.14}$$

if $a = 1$ or 2 assuming that $2\langle k \rangle + N + a - 3 \geq 0$. In this case one can define the (k, a)-generalized translation operator via an integral combining the inversion formulae of the (k, a)-generalized Fourier transform for $a = \frac{2}{n}$, $n \in \mathbb{N}$. For the general case of $2\langle k \rangle + N + a - 3 \geq 0$, the condition of boundedness (2.14) is not necessarily true. In [11], the authors proved such boundedness for $a = \frac{2}{n}$, $n \in \mathbb{N}$ only. And in [24], the authors found some negative results. They proved that if $a \in (1, 2) \cup (2, +\infty)$, then $\|B_{k,a}\|_\infty > 1$ (either finite or infinite). They also found the necessary and sufficient condition for the boundedness of the kernel $B_{k,a}(x, y)$ for the one dimensional case, i.e., $4\langle k \rangle + a - 2 \geq 0$.

3 Fractional Hardy inequalities

We will be interested in Hardy inequalities of the form

$$\int_X \frac{|f(x)|^2}{\left(1+|x|^2\right)^\sigma} d\eta(x) \leq B_\sigma \langle L^\sigma f, f\rangle \tag{3.1}$$

(or the Hardy inequality with homogeneous potential) for given $0 < \sigma < 1$, where L^σ is the fractional powers of a non-negative self-adjoint operator L and B_σ is a constant. It is a generalization of the classical Hardy inequality on $\mathbb{R}^N$

$$\frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{|f(x)|^2}{\|x\|^2} dx \leq \int_{\mathbb{R}^N} |\nabla f(x)|^2 \, dx, \ N \geq 3.$$

In [12], Ó. Ciaurri, L. Roncal and S. Thangavelu worked with *conformally invariant* fractional powers of Dunkl–Hermite operators $\mathbf{H}_k = -\triangle_k + \|x\|^2$, where $\triangle_k$ is the generalization of classical Laplacian on Euclidean space called Dunkl Laplacian, and proved the fractional Hardy inequalities for these operators of form (3.1) using ground state representation. The conformal invariant fractional powers was borrowed from the context of sublaplacians on Heisenberg groups (see [33]). They also deduced the Hardy inequalities for pure fractional powers of Dunkl–Hermite operators $\mathbf{H}_k^\sigma$ (see [12, Corollary 1.5]) as a consequence of the conformally invariant fractional Hardy inequalities.

We will prove a Hardy inequality of type (3.1) for fractional powers of the a-deformed Dunkl–Hermite operator $\triangle_{k,a} = \|x\|^{2-a} \triangle_k - \|x\|^a$ using the spherical harmonic expansion of the (k, a)-generalized Laguerre semigroup (3.8).

Theorem 3.1. *([41]) Let us define the constant*

$$B^\delta_{\alpha,\sigma} := \delta^\sigma \frac{\Gamma\left(\frac{\alpha+2+\sigma}{2}\right)}{\Gamma\left(\frac{\alpha+2-\sigma}{2}\right)}.$$

For $0 < \sigma < 1, \ \delta > 0$ *and* $4\langle k\rangle + 2N + a - 4 \geq 0$,

$$\left(\frac{a}{2}\right)^\sigma B^\delta_{\lambda_a,\sigma} \int_{\mathbb{R}^N} \frac{|f(x)|^2}{\left(\delta + \frac{2}{a}\|x\|^a\right)^\sigma} \vartheta_{k,a}(x) dx \leq \left\langle (-\triangle_{k,a})_\sigma f, f\right\rangle_{L^2\left(\mathbb{R}^N, \vartheta_{k,a}(x)dx\right)}$$

for all $f \in C_0^\infty(\mathbb{R}^N)$.

When $a = 2$, this inequality reduces to the fractional Hardy inequality in [12], which was proved using Dunkl–Hermite expansions. The definition of the modified fractional operator

$(-\triangle_{k,a})_\sigma$ will be given analogously as in [12] in Section 3.3. We can also deduce the Hardy inequalities for pure fractional powers of the operator $(-\triangle_{k,a})^\sigma$ analogous to Corollary 1.5 in [12] from this Hardy inequality. An uncertainty principle for fractional powers of $\triangle_{k,a}$ can also be deduced from this Hardy inequality as in [33].

There have also been several other studies of Hardy inequalities of form (3.1). For example, D. Gorbachev, V. Ivanov and S. Tikhonov [22] proved a sharp Pitt's inequality for Dunkl transform in $L^2\left(\mathbb{R}^N\right)$. Such Pitt's inequalities can imply a Hardy inequality of the form (3.1) for fractional powers of the Dunkl Laplacian $\triangle_k$. They also proved a sharp Pitt's inequality for the generalized Fourier transform $F_{k,a}$ in [23] using the Bochner-type identity (3.9), a particular case of the expansion (3.8) we will use. By the formula (5.6 b) in [8], The fractional powers of $-\|x\|^{2-a}\Delta_k$ can be naturally defined as follows,

$$F_{k,a}\left(\left(-\|\cdot\|^{2-a}\Delta_k\right)^\beta f\right)(\xi) = \left(\|\xi\|^a\right)^\beta F_{k,a}(f)(\xi).$$

And then from the inversion formula [8, Theorem 5.3] of the (k,a)-generalized Fourier transform, the Pitt's inequality in [23] implies also a Hardy inequality of the form (3.1) for $L = -\|x\|^{2-a}\Delta_k$ for $a = \frac{2}{n}$, $n \in \mathbb{N}_+$. When $a = 2$, this Hardy inequality reduces to that for fractional powers of the Dunkl Laplacian in [22]. The two Pitt's inequalities imply the logarithmic uncertainty principle for the Dunkl transform and $F_{k,a}$, respectively.

The results in this chapter are based on my paper [41]. In Section 3.1, we give the definitions of the a-deformed Laguerre convolution and the fractional a-deformed Laguerre operators, and then prove the radial Hardy inequality for the fractional a-deformed Laguerre operators. In Section 3.2 we give the proof of the spherical harmonic expansion of the (k,a)-generalized Laguerre semigroup and show that it reduces to the Bochner-type identity when z takes the boundary value $\frac{\pi i}{2}$. In Section 3.3, we give the proof of the fractional Hardy inequality in Theorem 3.1 using the expansion in Section 3.2. In Section 3.4 we study the relationship of the expansion in Section 3.2 with $\mathfrak{sl}_2$-representation. We will give a tangible characterization of the radial part of the (k,a)-generalized Laguerre semigroup on each k-spherical component $\mathcal{H}_k^m\left(\mathbb{R}^N\right)$ for $\lambda_{k,a,m} \geq -1/2$.

3.1 The a-deformed Laguerre operator

The Laguerre translation $\mathcal{T}_r^\alpha$ was introduced by McCully [29] for $\alpha = 0$ and was extended to $\alpha \geq -1/2$ (see [3] or [42, Chapter 6]). We define the *a-deformed Laguerre translation* as

$$\mathcal{T}_r^{a,\alpha} f(s) := \frac{\Gamma(\alpha+1)2^\alpha}{\sqrt{2\pi}} \int_0^\pi f\left(\left(r^a + s^a + 2r^{\frac{a}{2}} s^{\frac{a}{2}} \cos\theta\right)^{1/a}\right) \cdot \\ J_{\alpha-1/2}\left(\frac{2}{a} r^{\frac{a}{2}} s^{\frac{a}{2}} \sin\theta\right) \left(\frac{2}{a} r^{\frac{a}{2}} s^{\frac{a}{2}} \sin\theta\right)^{-(\alpha-1/2)} (\sin\theta)^{2\alpha}\, d\theta$$

for $r, s > 0$ and $\alpha \geq -1/2$, where J_ν is the Bessel function of order ν. When $a = 2$, it reduces to the Laguerre translation $\mathcal{T}_r^\alpha$ in [12]. The results in [12] are also valid for the critical case when $\alpha = -1/2$ since the definition of the Laguerre translation can be extended to this case. If f and g are functions defined on $(0, \infty)$, the *a-deformed Laguerre convolution* $f *_{a,\alpha} g$ is given by

$$f *_{a,\alpha} g(r) = \int_0^\infty \mathcal{T}_r^{a,\alpha} f(s) g(s) s^{a\alpha+a-1}\, ds. \tag{3.2}$$

By changing variables

$$r = \left(\frac{a}{2}\right)^{1/a} r_1^{2/a}, \quad s = \left(\frac{a}{2}\right)^{1/a} s_1^{2/a}$$

and setting

$$f_1 = f\left(\left(\frac{a}{2}\right)^{1/a} (\cdot)^{2/a}\right), \quad g_1 = g\left(\left(\frac{a}{2}\right)^{1/a} (\cdot)^{2/a}\right),$$

we have

$$\begin{aligned}\int_0^\infty \mathcal{T}_r^{a,\alpha} f(s) g(s) s^{a\alpha+a-1}\, ds &= \left(\frac{a}{2}\right)^{\alpha+1} \int_0^\infty \mathcal{T}_{r_1}^{\alpha} f_1(s_1) g_1(s_1) s_1^{2\alpha+1}\, ds_1 = \left(\frac{a}{2}\right)^{\alpha+1} f_1 *_\alpha g_1(r_1) \\ &= \left(\frac{a}{2}\right)^{\alpha+1} g_1 *_\alpha f_1(r_1) = \left(\frac{a}{2}\right)^{\alpha+1} \int_0^\infty \mathcal{T}_{r_1}^{\alpha} g_1(s_1) f_1(s_1) s_1^{2\alpha+1}\, ds_1 \\ &= \int_0^\infty \mathcal{T}_r^{a,\alpha} g(s) f(s) s^{a\alpha+a-1}\, ds,\end{aligned}$$

where $f *_\alpha g$ is the Laguerre convolution defined in [42, Chapter 6]. Thus $f *_{a,\alpha} g(r) = g *_{a,\alpha} f(r)$.

Let

$$\varphi_l^{a,\alpha}(r) := L_l^\alpha\left(\frac{2}{a} r^a\right) \exp\left(-\frac{1}{a} r^a\right), \quad l = 0, 1, \cdots.$$

Then substituting r as $\sqrt{\frac{2}{a}} r^{\frac{a}{2}}$ and s as $\sqrt{\frac{2}{a}} s^{\frac{a}{2}}$ in the formula (3.2) in [12], we get

$$\mathcal{T}_r^{a,\alpha} \varphi_n^{a,\alpha}(s) = \frac{n!}{(\alpha+1)_n} \varphi_n^{a,\alpha}(r) \varphi_n^{a,\alpha}(s), \qquad \alpha \geq -1/2. \tag{3.3}$$

The Laguerre operator

$$L_\alpha = -\frac{d^2}{dr^2} + r^2 - \frac{2\alpha+1}{r}\frac{d}{dr} \tag{3.4}$$

studied in [12] is a symmetric operator on $L^2\left((0,\infty), d\mu_\alpha\right)$, where $\alpha \geq -1/2$ and $d\mu_\alpha(r) = r^{2\alpha+1}dr$. The functions

$$\widetilde{\varphi}_l^\alpha(r) = \left(\frac{2\Gamma(l+1)}{\Gamma(\alpha+l+1)}\right)^{1/2} L_l^\alpha\left(r^2\right)\exp\left(-\frac{1}{2}r^2\right),\ l = 0, 1, \cdots$$

are eigenfunctions of L_α with eigenvalues $2\left(2l+\alpha+1\right)$.

Substituting r by $u = \sqrt{\frac{2}{a}}r^{\frac{a}{2}}$ in (3.4),

$$\begin{aligned} -\frac{d^2}{du^2} + u^2 - \frac{2\alpha+1}{u}\frac{d}{du} &= -\frac{2}{a}\left(\frac{1}{r^{a-2}}\frac{d^2}{dr^2} + \left(1-\frac{a}{2}\right)\frac{1}{r^{a-1}}\frac{d}{dr}\right) + \frac{2}{a}r^a - \frac{2\alpha+1}{r^{a-1}}\frac{d}{dr} \\ &= \frac{2}{a}\left(-\frac{1}{r^{a-2}}\frac{d^2}{dr^2} + r^a - (a\alpha+1)\frac{1}{r^{a-1}}\frac{d}{dr}\right). \end{aligned}$$

The a-deformed Laguerre differential operator can then be defined as

$$L_{a,\alpha} = -\frac{1}{r^{a-2}}\frac{d^2}{dr^2} + r^a - (a\alpha+1)\frac{1}{r^{a-1}}\frac{d}{dr}. \tag{3.5}$$

It is symmetric on $L^2(0,\infty)$ with respect to the measure $d\mu_{a,\alpha}(r) = r^{a\alpha+a-1}dr$, $\alpha \geq -1/2$. When $a = 2$, the operator reduces to the Laguerre operator (3.4).

Define the Laguerre functions of type α as

$$\widetilde{\varphi}_l^{a,\alpha}(r) = \left(\frac{2^{\alpha+1}\Gamma(l+1)}{a^\alpha\Gamma(\alpha+l+1)}\right)^{1/2} L_l^\alpha\left(\frac{2}{a}r^a\right)\exp\left(-\frac{1}{a}r^a\right),\ l = 0, 1, \cdots,$$

where $\alpha \geq -1/2$. Then they form an orthonormal basis of $L^2\left((0,\infty), d\mu_{a,\alpha}\right)$ (this is also the case of Proposition 2.1 when $\alpha = \lambda_{k,a,m}$) and are the eigenfunctions of the a-deformed Laguerre operator (3.5). Indeed,

$$L_{a,\alpha}\widetilde{\varphi}_l^{a,\alpha} = a\left(2l+\alpha+1\right)\widetilde{\varphi}_l^{a,\alpha},\quad l = 0, 1, \cdots.$$

It suffices to substitute r by $\sqrt{\frac{2}{a}}r^{\frac{a}{2}}$ in the conclusions of [12, Section 3] to get this.

The Laguerre expansion of $f \in L^2((0,\infty), d\mu_{a,\alpha})$, namely the expansion

$$f = \sum_{l=0}^{\infty}\left(\frac{2^{\alpha+1}\Gamma(l+1)}{a^\alpha\Gamma(\alpha+l+1)}\right)\langle f, \varphi_l^{a,\alpha}\rangle_{d\mu_{a,\alpha}}\varphi_l^{a,\alpha}$$

can be written in a compact form in terms of Laguerre convolution.

Lemma 3.2. *For a function $f \in L^2((0,\infty), d\mu_{a,\alpha})$, $\varphi_l^{a,\alpha}$ is an eigenfunction of f, i.e.,*

$$f *_{a,\alpha} \varphi_l^{a,\alpha} = \frac{\Gamma(\alpha+1)\Gamma(l+1)}{\Gamma(\alpha+l+1)} \langle f, \varphi_l^{a,\alpha}\rangle_{d\mu_{a,\alpha}} \varphi_l^{a,\alpha}.$$

In particular,

$$\delta_{nj}\varphi_n^{a,\alpha} = \frac{2^{\alpha+1}}{a^\alpha \Gamma(\alpha+1)} \varphi_n^{a,\alpha} *_{a,\alpha} \varphi_j^{a,\alpha}. \tag{3.6}$$

Proof. Omitted. It is only a slight modification of the proof of Lemma 3.1 in [12]. □

Thus $f *_{a,\alpha} \varphi_l^{a,\alpha}$ are eigenfunctions of $L_{a,\alpha}$ with the eigenvalues $a\,(2l+\alpha+1)$ for $l = 0, 1, \cdots$ and we have the spectral decomposition of the a-deformed Laguerre operator

$$L_{a,\alpha} f = \frac{2^{\alpha+1}}{a^\alpha \Gamma(\alpha+1)} \sum_{l=0}^{\infty} a\,(2l+\alpha+1)\, f *_{a,\alpha} \varphi_l^{a,\alpha}.$$

It is then natural to define fractional powers of Laguerre operators as

$$L_{a,\alpha}^\sigma f = \frac{2^{\alpha+1}}{a^\alpha \Gamma(\alpha+1)} \sum_{l=0}^{\infty} (a\,(2l+\alpha+1))^\sigma\, f *_{a,\alpha} \varphi_l^{a,\alpha}, \quad \alpha \geq -1/2.$$

But it suits better to work with the modified fractional operator $L_{a,\alpha;\sigma}$ with the spectrum $4^\sigma S_l^{a,\alpha;\sigma}$, i.e.,

$$L_{a,\alpha,\sigma} f = \frac{2^{\alpha+1}}{a^\alpha \Gamma(\alpha+1)} \sum_{l=0}^{\infty} (2a)^\sigma S_l^{a,\alpha;\sigma} f *_{a,\alpha} \varphi_l^{a,\alpha}, \quad \alpha \geq -1/2,$$

where

$$S_l^{a,\alpha;\sigma} = \frac{\Gamma\left(\frac{a(2l+\alpha+1)}{2a} + \frac{1+\sigma}{2}\right)}{\Gamma\left(\frac{a(2l+\alpha+1)}{2a} + \frac{1-\sigma}{2}\right)},$$

because such fractional powers of the operator correspond to the conformally invariant fractional powers of sublaplacian $\mathcal{L}$ on Heisenberg groups when we consider the conformally invariant fractional powers $\mathcal{L}_\sigma$ (see [33]) acting on the functions of the form $e^{it} f(|z|)$. In short, we write

$$L_{a,\alpha;\sigma} = (2a)^\sigma \frac{\Gamma\left(\frac{L_{a,\alpha}}{2a} + \frac{1+\sigma}{2}\right)}{\Gamma\left(\frac{L_{a,\alpha}}{2a} + \frac{1-\sigma}{2}\right)}.$$

The motivation for this definition goes back to [9, (1.33)], for instance.

For $\delta > 0$ and $\alpha \geq -1/2$, denote

$$\omega_{\alpha,\sigma}^{\delta,a}(r) := c_{\alpha,\sigma} \left(\delta + \tfrac{2}{a} r^a\right)^{-(\alpha+1+\sigma)/2} K_{(\alpha+1+\sigma)/2}\left(\frac{\delta + \frac{2}{a} r^a}{2}\right),$$

where K_ν is the Macdonald's function of order ν (see [28, Chapter 5, Section 5.7]), and $c_{\alpha,\sigma}$ is the constant

$$c_{\alpha,\sigma} := \frac{\sqrt{\pi}2^{1-\sigma}}{\Gamma\left((\alpha+2+\sigma)/2\right)}.$$

In [12], the authors proved a Hardy inequality for the fractional Laguerre operator for the case of $a=2$ using ground state representation.

Theorem 3.3. *([12, Theorem 1.1]) Let $0<\sigma<1$, $\delta>0$, and $2\alpha+1>0$. Then*

$$B^\delta_{\alpha,\sigma}\int_0^\infty \frac{|f(r)|^2}{(\delta+r^2)^\sigma}d\mu_\alpha(r) \le \frac{4^\sigma}{\delta^\sigma}\left(B^\delta_{\alpha,\sigma}\right)^2\int_0^\infty |f(r)|^2\frac{\omega^\delta_{\alpha,\sigma}(r)}{\omega^\delta_{\alpha,-\sigma}(r)}d\mu_\alpha(r) \le \left\langle L_{\alpha,\sigma}f,f\right\rangle_{d\mu_\alpha}$$

for all $f\in C_0^\infty(0,\infty)$.

Taking f as the Laguerre functions for the case of $a=2$, and then substituting r by $\sqrt{\frac{2}{a}}r^{\frac{a}{2}}$, we get

$$B^\delta_{\alpha,\sigma}\int_0^\infty \frac{|\widetilde{\varphi}^{a,\alpha}_l(r)|^2}{\left(\delta+\dfrac{2}{a}r^a\right)^\sigma}d\mu_{a,\alpha}(r) \le \frac{4^\sigma}{\delta^\sigma}\left(B^\delta_{\alpha,\sigma}\right)^2\int_0^\infty |\widetilde{\varphi}^{a,\alpha}_l(r)|^2\frac{\omega^{\delta,a}_{\alpha,\sigma}(r)}{\omega^{\delta,a}_{\alpha,-\sigma}(r)}d\mu_{a,\alpha}(r)$$
$$\le \left\langle \left(\frac{2}{a}L_{a,\alpha}\right)_\sigma \widetilde{\varphi}^{a,\alpha}_l, \widetilde{\varphi}^{a,\alpha}_l\right\rangle_{d\mu_{a,\alpha}}$$

for $\alpha\ge -1/2$. Here $\left(\frac{2}{a}L_{a,\alpha}\right)_\sigma = 4^\sigma\frac{\Gamma\left(\frac{\frac{2}{a}L_{a,\alpha}}{4}+\frac{1+\sigma}{2}\right)}{\Gamma\left(\frac{\frac{2}{a}L_{a,\alpha}}{4}+\frac{1-\sigma}{2}\right)}$ and it equals to $\left(\frac{2}{a}\right)^\sigma L_{a,\alpha;\sigma}$.

Then using the expansion via Laguerre functions, we derive the Hardy inequality for the fractional a-deformed Laguerre operator.

Theorem 3.4. *([41]) Let $0<\sigma<1$, $\delta>0$, and $\alpha\ge -1/2$. Then*

$$\left(\frac{a}{2}\right)^\sigma B^\delta_{\alpha,\sigma}\int_0^\infty \frac{|f(r)|^2}{\left(\delta+\dfrac{2}{a}r^a\right)^\sigma}d\mu_{a,\alpha}(r) \le \left(\frac{2a}{\delta}\right)^\sigma\left(B^\delta_{\alpha,\sigma}\right)^2\int_0^\infty |f(r)|^2\frac{\omega^{\delta,a}_{\alpha,\sigma}(r)}{\omega^{\delta,a}_{\alpha,-\sigma}(r)}d\mu_{a,\alpha}(r)$$
$$\le \left\langle L_{a,\alpha;\sigma}f,f\right\rangle_{d\mu_{a,\alpha}}$$

for all $f\in C_0^\infty(0,\infty)$.

3.2 Spherical harmonic expansion of the (k,a)-generalized Laguerre semigroup

The holomorphic semigroup related to the a-deformed Laguerre operator $L_{a,\alpha}$ is defined on $L^2((0,\infty), d\mu_{a,\alpha})$ by

$$I_{a,\alpha;z}f = e^{-\frac{z}{a}L_{a,\alpha}}f, \quad \Re z\ge 0. \tag{3.7}$$

From the spectral decomposition of $L_{a,\alpha}$ it equals to

$$\frac{2^{\alpha+1}}{a^{\alpha}\Gamma(\alpha+1)}\sum_{l=0}^{\infty}e^{-z(2l+\alpha+1)}f*_{a,\alpha}\varphi_l^{a,\alpha}.$$

We will show that this a-deformed Laguerre holomorphic semigroup reduces to a-deformed Hankel transform $H_{a,\alpha}$ when taking the boundary value $z=\frac{\pi i}{2}$. The operators $L_{a,\alpha}$ also give an explicit expression of the radial part $\Omega_{k,a}^{(m)}(\gamma_z)$ of the (k,a)-generalized Laguerre semigroup on each k-spherical component $\mathcal{H}_k^m\left(\mathbb{R}^N\right)$ defined via decomposition of unitary representation in [8, Section 4.1], i.e., $\Omega_{k,a}^{(m)}(\gamma_z)f(s)=s^mI_{a,\lambda_{k,a,m};z}\left((\cdot)^{-m}f\right)(s)$, $\Re z\geq 0$, $s>0$ for $f\in L^2\left(\mathbb{R}_+,r^{\,2\langle k\rangle+N+a-3}dr\right)$ and $\lambda_{k,a,m}\geq -1/2$ as will be shown in Section 3.4. We denote $\lambda_a:=\frac{2\langle k\rangle+N-2}{a}$.

Theorem 3.5. *([41]) For any function $f\in L^2\left((0,\infty),d\mu_{a,\alpha}\right)$, $\alpha\geq -1/2$, we have*

$$e^{(\alpha+1)\pi i/2}I_{a,\alpha;\frac{\pi i}{2}}(f)=H_{a,\alpha}(f),$$

where the a-deformed Hankel transform is defined as

$$H_{a,\alpha}(f)(r)=\frac{1}{a^{\alpha}\Gamma(\alpha+1)}\int_0^{\infty}f(s)j_{\alpha}\left(\frac{2}{a}r^{\frac{a}{2}}s^{\frac{a}{2}}\right)s^{a\alpha+a-1}\,ds$$

and $j_\alpha(t)=2^{\alpha}\Gamma(\alpha+1)t^{-\alpha}J_\alpha(t)$ is the normalized Bessel function.

We will then give a spherical harmonic expansion of the (k,a)-generalized Laguerre semigroup.

Theorem 3.6. *([41])(Spherical harmonic expansion of the (k,a)-generalized Laguerre semigroup) For $f\in L^2\left(\mathbb{R}^N,\vartheta_{k,a}(x)\,dx\right)$, $4\langle k\rangle+2N+a-4\geq 0$, and $x\in\mathbb{R}^N$, $x=rx'$, with $r\in\mathbb{R}^+$, $x'\in\mathbb{S}^{N-1}$, we have*

$$\mathcal{I}_{k,a}(z)f(x)=\sum_{m,j}Y_{m,j}(x')r^mI_{a,\lambda_{k,a,m};z}\left((\cdot)^{-m}f_{m,j}\right)(r),\tag{3.8}$$

where $\Re z\geq 0$. Specially, the (k,a)-generalized Laguerre semigroup reduces to the one dimensional a-deformed Laguerre holomorphic semigroup for radial functions, that is, for $f=f_0(\|\cdot\|)$, $f_0\in L^2\left(\mathbb{R}_+,r^{\,2\langle k\rangle+N+a-3}dr\right)$ and $r=\|x\|$, we have

$$\mathcal{I}_{k,a}(z)f(x)=\left(\mathcal{I}_{k,a}(z)f\right)_0(r),\quad\left(\mathcal{I}_{k,a}(z)f\right)_0(r)=I_{a,\lambda_a;z}(f_0)(r).$$

Remark 3.7. *i). This theorem, together with* (1.1) *and Theorem 3.5, imply the Bochner-type identity in [8, Theorem 5.21], which was used in [23] for Schwartz functions to prove Pitt's inequalities for the generalized Fourier transform. That is, taking the boundary value* $z = \frac{\pi i}{2}$*, the expansion reduces to*

$$F_{k,a} f(x) = \sum_{m,j} e^{-i\pi m/a} Y_{m,j}(x') r^m H_{a,\lambda_{k,a,m}} \left((\cdot)^{-m} f_{m,j} \right)(r) . \tag{3.9}$$

This theorem also generalizes the result in [43] that Hermite semigroups reduce to Laguerre semigroups of type $\frac{N}{2} - 1$ *(the case of* $a = 2$ *and* $k = 0$*) for radial functions on* $\mathbb{R}^N$*.*
ii). When $a = 2$ *and* $z = 2t$*,* $t > 0$*, the expansion reduces to the formula given in Theorem 4.5 in [12], but our proof is different from that in [12] even in this case because we used the new tools introduced by S. Ben Saïd, T. Kobayashi and B. Ørsted [8] in the development of* (k, a)*-generalized Fourier analysis.*

Proof of Theorem 3.5.

Define

$$q_{a,\alpha;z}(r) := \frac{2^{\alpha+1}}{a^\alpha \Gamma(\alpha+1)} \sum_{l=0}^\infty e^{-z(2l+\alpha+1)} \varphi_l^{a,\alpha}(r) = \left(\frac{2}{a}\right)^\alpha q_{2,\alpha;z}\left(\sqrt{\frac{2}{a}} r^{\frac{a}{2}}\right).$$

Then we can write

$$e^{-\frac{z}{a} L_{a,\alpha}} f = f *_{a,\alpha} q_{a,\alpha;z}.$$

We give the kernel of the holomorphic semigroup $I_{a,\alpha;z}$.

Lemma 3.8. *Let* $\alpha \geq -1/2$*,* $\Re z \geq 0$ *and* $z \neq 0$*, we have that*

$$\mathcal{T}_r^{a,\alpha} q_{a,\alpha;z}(s) = \frac{e^{-\frac{\coth z}{a}(r^a + s^a)}}{(r^{\frac{a}{2}} s^{\frac{a}{2}})^\alpha \sinh z} I_\alpha \left(\frac{\frac{2}{a} r^{\frac{a}{2}} s^{\frac{a}{2}}}{\sinh z} \right),$$

where I_α *is the modified Bessel function of the first kind and order* α*, see [28, Chapter 5, Section 5.7].*

Proof. For the case when $a = 2$, we take $w = e^{-z}$ in the equality (see [42, p. 83])

$$\sum_{n=0}^\infty \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} \varphi_n^\alpha(r) \varphi_n^\alpha(s) w^{2n}$$
$$= (1-w^2)^{-1} (rsw)^{-\alpha} \exp\Big\{ -\frac{1}{2}\Big(\frac{1+w^2}{1-w^2}\Big)(r^2+s^2)\Big\} I_\alpha\Big(\frac{2wrs}{1-w^2}\Big), |w| < 1.$$

Then we get the Lemma for $a = 2$. And it reduces to Lemma 3.2 in [12] when $z = 2t,\ t > 0$ in this case.

For the general case of $a > 0$, change variables

$$r = \left(\frac{a}{2}\right)^{1/a} r_1^{2/a}, \quad s = \left(\frac{a}{2}\right)^{1/a} s_1^{2/a}.$$

Then we get

$$\begin{aligned}\mathcal{T}_r^{a,\alpha} q_{a,\alpha;z}(s) &= \left(\frac{2}{a}\right)^{\alpha} \mathcal{T}_{r_1}^{\alpha} q_{2,z;\alpha}(s_1)\\ &= \left(\frac{2}{a}\right)^{\alpha} \frac{e^{-\frac{\coth\ z}{2}(r_1^2+s_1^2)}}{(r_1 s_1)^{\alpha}\sinh\ z} I_\alpha\left(\frac{r_1 s_1}{\sinh\ z}\right) = \frac{e^{-\frac{\coth\ z}{2}\frac{2}{a}(r^a+s^a)}}{(r^{\frac{a}{2}} s^{\frac{a}{2}})^{\alpha}\sinh\ z} I_\alpha\left(\frac{\frac{2}{a} r^{\frac{a}{2}} s^{\frac{a}{2}}}{\sinh\ z}\right).\end{aligned}$$

The proof of Lemma 3.8 is therefore completed. This Lemma can also be deduced from Hille–Hardy identity directly. □

Let $z = i\frac{\pi}{2}$. Then from formula (5.7.4) in [28],

$$I_\alpha\left(\frac{\frac{2}{a} r^{\frac{a}{2}} s^{\frac{a}{2}}}{\sinh i\frac{\pi}{2}}\right) = e^{-\alpha\pi i/2} J_\alpha\left(\frac{2}{a} r^{\frac{a}{2}} s^{\frac{a}{2}}\right) = e^{-\alpha\pi i/2} \frac{\left(\frac{2}{a} r^{\frac{a}{2}} s^{\frac{a}{2}}\right)^{\alpha}}{2^{\alpha}\Gamma(\alpha+1)} j_\alpha\left(\frac{2}{a} r^{\frac{a}{2}} s^{\frac{a}{2}}\right).$$

So

$$\begin{aligned}I_{a,\alpha;i\frac{\pi}{2}} f(r) &= f *_{a,\alpha} q_{a,\alpha;i\frac{\pi}{2}}(r) = \int_0^\infty f(s)\mathcal{T}_r^{a,\alpha} q_{a,\alpha;i\frac{\pi}{2}}(s) s^{a\alpha+a-1}\,ds\\ &= e^{-(\alpha+1)\pi i/2} \frac{1}{a^{\alpha}\Gamma(\alpha+1)} \int_0^\infty f(s) j_\alpha\left(\frac{2}{a} r^{\frac{a}{2}} s^{\frac{a}{2}}\right) s^{a\alpha+a-1}\,ds = e^{-(\alpha+1)\pi i/2} H_{a,\alpha}(f)(r).\end{aligned}$$

The proof of Theorem 3.5 is therefore completed. □

Consider the orthonormal basis (2.2) of $\mathcal{H}_k^m\left(\mathbb{R}^N\right)|_{\mathbb{S}^{N-1}}$. Accordingly, we have the h-harmonic expansion for $f \in L^2\left(\mathbb{R}^N, \vartheta_{k,a}(x)\,dx\right)$,

$$f(rx') = \sum_{m=0}^{\infty} \sum_{i=1}^{d(m)} f_{m,i}(r) Y_i^m(x'), \tag{3.10}$$

where

$$f_{m,i}(r) = \int_{\mathbb{S}^{N-1}} f(rx')\, Y_i^m(x') \vartheta_{k,a}(x') d\sigma(x').$$

Proof of Theorem 3.6. By Lemma 3.1, the a-deformed Laguerre holomorphic semigroup can also be written as

$$I_{a,\alpha;z} f = \sum_{l=0}^{\infty} e^{-z(2l+\alpha+1)} \langle f, \widetilde{\varphi}_l^{a,\alpha}\rangle_{d\mu_{a,\alpha}} \widetilde{\varphi}_l^{a,\alpha}.$$

We then apply the spherical harmonic expansion (3.10) to the spectral definition (2.9) of $\mathcal{I}_{k,a}(z)f(x)$. By (2.4) and by noticing that

$$\widetilde{\varphi}_l^{a,\lambda_{k,a,m}}(r) = r^{-m}\psi_{l,m}^{(a)}(r)$$

when $\lambda_{k,a,m} \geq -1/2$, we have

$$\begin{aligned}
\mathcal{I}_{k,a}(z)f(x) &= \sum_{l,m,j} e^{-z\left(2l+\lambda_{k,a,m}+1\right)} \left\langle f, \Phi_{l,m,j}^{(a)} \right\rangle_{k,a} \Phi_{l,m,j}^{(a)}(x) \\
&= \sum_{m=0}^{\infty}\sum_{j=0}^{d(m)}\sum_{l=0}^{\infty} \int_0^\infty f_{m,j}(r)\,\psi_{l,m}^{(a)}(r)\, r^{2\langle k\rangle+N+a-3} dr \cdot \\
&\quad e^{-z\left(2l+\lambda_{k,a,m}+1\right)}\psi_{l,m}^{(a)}(r)\, Y_{m,j}(x') \\
&= \sum_{m=0}^{\infty}\sum_{j=0}^{d(m)}\sum_{l=0}^{\infty} \int_0^\infty f_{m,j}(r)\, r^{-m}\widetilde{\varphi}_l^{a,\lambda_{k,a,m}}(r)\, r^{a\lambda_{k,a,m}+a-1} dr \cdot \\
&\quad e^{-z\left(2l+\lambda_{k,a,m}+1\right)}\widetilde{\varphi}_l^{a,\lambda_{k,a,m}}(r)\, r^m Y_{m,j}(x') \\
&= \sum_{m,j} Y_{m,j}(x') r^m I_{a,\lambda_{k,a,m};z}\left((\cdot)^{-m} f_{m,j}\right)(r).
\end{aligned}$$

For $f(x) = Y_{m,j}(x')\,\psi(r),\ \psi(r) \in L^2\left(\mathbb{R}_+, r^{\,2\langle k\rangle+N+a-3}dr\right),\ x = rx'$, we have the following Hecke-Bochner identity for the (k,a)-generalized Laguerre semigroups,

$$\mathcal{I}_{k,a}(z)f(x) = Y_{m,j}(x') r^m I_{a,\lambda_{k,a,m};z}\left((\cdot)^{-m}\psi\right)(r).$$

Taking $m = 0$, we get the special case for radial functions. The proof of Theorem 3.6 is therefore completed. □

Define the a-deformed Dunkl–Hermite heat semigroup with infinitesimal generator $\Delta_{k,a}$ as $T_t^{k,a}f := \mathcal{I}_{k,a}(ta)f,\ t > 0$ and the a-deformed Laguerre heat semigroup as $T_{a,\alpha;t}f := I_{a,\alpha;ta}f,\ t > 0$. Then from Theorem 3.6,

$$T_t^{k,a}f(x) = \sum_{m,j} Y_{m,j}(x') r^m T_{a,\lambda_{k,a,m},t}\left((\cdot)^{-m} f_{m,j}\right)(r). \tag{3.11}$$

It reduces to the equation in Theorem 4.5 in [12] when $a = 2$.

Remark 3.9. *The case of $a = 2$ of the above argument gives a new proof of the Theorem 4.5 in [12]. In [12] the authors proved the Theorem 4.5 by using Dunkl–Hermite expansions and proving the identity for Dunkl–Hermite projections first. But if we use the basis given in* *terms of Laguerre polynomials, which are also the eigenfunctions of Dunkl Hermite operators, the theorem can be proven directly from the above. For radial functions it was shown in [43] in classical case that Hermite expansions reduce to Laguerre expansions. The Heisenberg uncertainty principle for Dunkl transforms was also proved using the two different expansions successively. It was first proved by Rösler using Dunkl–Hermite expansions (see [34]), and was then proved in [8, Section 5.7] using the tools we refer to in this paper as well (see [37] also for a proof using the basis given by Dunkl [20] in terms of Laguerre polynomials).*

3.3 Proof of Theorem 3.1

Now we use the following Lemma (see [12]) to give the expansion of the fractional (k,a)-generalized harmonic ocillator into fractional a-deformed Laguerre operator (there is a constant missed in [12, Lemma 3.4]. Here we give the corrected Lemma).

Lemma 3.10. *([12, Lemma 3.4]) Let $0<\sigma<1$, and $\lambda\in\mathbb{R}$ such that $\lambda+\sigma>-1$. Then,*

$$2^{\sigma}\left|\Gamma\left(-\sigma\right)\right|\frac{\Gamma\left(\frac{\lambda}{2}+\frac{1+\sigma}{2}\right)}{\Gamma\left(\frac{\lambda}{2}+\frac{1-\sigma}{2}\right)}=\int_0^{\infty}\left(\cosh\ t-1\right)\left(\sinh\ t\right)^{-\sigma-1}dt+\int_0^{\infty}\left(1-e^{-t\lambda}\right)\left(\sinh\ t\right)^{-\sigma-1}dt.$$

Denote by $E_\sigma:=\frac{a^\sigma}{|\Gamma(-\sigma)|}\int_0^\infty\left(\cosh\ t-1\right)\left(\sinh\ t\right)^{-\sigma-1}dt$. Then

$$\begin{aligned}L_{a,\alpha;\sigma}f\left(r\right)&=\frac{2^{\alpha+1}}{a^{\alpha}\Gamma(\alpha+1)}\sum_{l=0}^{\infty}(2a)^{\sigma}S_l^{a,\alpha;\sigma}f*_{a,\alpha}\varphi_l^{a,\alpha}(r)\\&=E_\sigma f\left(r\right)+\frac{a^\sigma}{\left|\Gamma\left(-\sigma\right)\right|}\int_0^\infty\left(f\left(r\right)-T_{a,\alpha;t/a}f\left(r\right)\right)\left(\sinh\ t\right)^{-\sigma-1}dt.\end{aligned}$$

Given $0<\sigma<1$, we define conformally invariant fractional (k,a)-generalized harmonic oscillator $\left(-\triangle_{k,a}\right)_\sigma$ to be the operator

$$\left(-\triangle_{k,a}\right)_\sigma=(2a)^\sigma\frac{\Gamma\left(\frac{-\Delta_{k,a}}{2a}+\frac{1+\sigma}{2}\right)}{\Gamma\left(\frac{-\Delta_{k,a}}{2a}+\frac{1-\sigma}{2}\right)}.$$

So, in view of (2.6), $\left(-\triangle_{k,a}\right)_\sigma$ corresponds to the spectral multiplier $(2a)^\sigma\Gamma\left(\frac{2l+\lambda_{k,a,m}+1}{2}+\frac{1+\sigma}{2}\right)/\Gamma\left(\frac{2l+\lambda_{k,a,m}+1}{2}+\frac{1-\sigma}{2}\right)$ and it equals to

$$E_\sigma f\left(x\right)+\frac{a^\sigma}{\left|\Gamma\left(-\sigma\right)\right|}\int_0^\infty\left(f\left(x\right)-T_{t/a}^{k,a}f\left(x\right)\right)\left(\sinh\ t\right)^{-\sigma-1}dt$$

from Lemma 3.10. For $a=2$, it should coincide with the fractional Dunkl–Hermite operator in [12] (there is a constant factor missed in the definition given in [12]).

By formula (3.11),

$$
\begin{aligned}
(-\triangle_{k,a})_{\sigma} f(x) &= E_{\sigma} f(x) + \frac{a^{\sigma}}{|\Gamma(-\sigma)|} \int_0^{\infty} \Big(f(x) - T^{k,a}_{t/a} f(x) \Big) (\sinh\ t)^{-\sigma-1}\, dt \\
&= \sum_{m,j} Y_{m,j}(x') r^m \Big[E_{\sigma} r^{-m} f_{m,j}(r) \\
&\quad + \frac{a^{\sigma}}{|\Gamma(-\sigma)|} \int_0^{\infty} \left(r^{-m} f_{m,j}(r) - T_{a,\lambda_{k,a,m},t/a} \left((\cdot)^{-m} f_{m,j} \right)(r) \right) (\sinh\ t)^{-\sigma-1}\, dt \Big] \\
&= \sum_{m,j} Y_{m,j}(x') r^m \Big[E_{\sigma} g_{m,j}(r) \\
&\quad + \frac{a^{\sigma}}{|\Gamma(-\sigma)|} \int_0^{\infty} \left(g_{m,j}(r) - T_{a,\lambda_{k,a,m},t/a} g_{m,j}(r) \right) (\sinh\ t)^{-\sigma-1}\, dt \Big] \\
&= \sum_{m,j} Y_{m,j}(x') r^m L_{a,\lambda_{k,a,m};\sigma} g_{m,j}(r),
\end{aligned}
$$

where $g_{m,j}(r) = r^{-m} f_{m,j}(r)$.

The following Lemma was found by Yafaev [46] for $v = m/2,\ m \in \mathbb{N}$, and was then proved in [23] for any $v > 0$.

Lemma 3.11. *([23, Lemma 2.3]) If* $v > 0$, *then*

$$
\frac{\Gamma(t+v)}{\Gamma(\tau+v)} < \frac{\Gamma(t)}{\Gamma(\tau)}, \qquad 0 < t < \tau.
$$

We can then start to prove the fractional Hardy inequality in Theorem 3.1 using the above expansion and lemma. By Theorem 3.4 we have

$$
\begin{aligned}
\left\langle (-\triangle_{k,a})_{\sigma} f, f \right\rangle_{L^2\left(\mathbb{R}^N, \vartheta_{k,a}(x)dx\right)} &= \sum_{m=0}^{\infty} \sum_{j=1}^{d(m)} \left\langle L_{a,\lambda_{k,a,m};\sigma} g_{m,j}, g_{m,j} \right\rangle_{L^2\left((0,\infty), d\mu_{a,\lambda_{k,a,m}}(r)\right)} \\
&\geq \sum_{m=0}^{\infty} \sum_{j=1}^{d(m)} \left(\frac{2a}{\delta} \right)^{\sigma} \left(B^{\delta}_{\lambda_{k,a,m},\sigma} \right)^2 \int_0^{\infty} |g_{m,j}(r)|^2 \frac{\omega^{\delta,a}_{\lambda_{k,a,m},\sigma}(r)}{\omega^{\delta,a}_{\lambda_{k,a,m},-\sigma}(r)} d\mu_{a,\lambda_{k,a,m}}(r) \\
&= \sum_{m=0}^{\infty} \sum_{j=1}^{d(m)} \left(\frac{2a}{\delta} \right)^{\sigma} \left(B^{\delta}_{\lambda_{k,a,m},\sigma} \right)^2 \int_0^{\infty} |f_{m,j}(r)|^2 \frac{\omega^{\delta,a}_{\lambda_{k,a,m},\sigma}(r)}{\omega^{\delta,a}_{\lambda_{k,a,m},-\sigma}(r)} d\mu_{a,\lambda_a}(r).
\end{aligned}
$$

Then by Lemma 3.11 and a similar argument as in the end of the proof in [12],

$$
\begin{aligned}
&\left(\frac{2a}{\delta} \right)^{\sigma} \left(B^{\delta}_{\lambda_{k,a,m},\sigma} \right)^2 \frac{\omega^{\delta,a}_{\lambda_{k,a,m},\sigma}(r)}{\omega^{\delta,a}_{\lambda_{k,a,m},-\sigma}(r)} \\
&\qquad = \left(\frac{a}{2} \right)^{\sigma} \delta^{\sigma} \frac{\Gamma\left(\frac{\lambda_{k,a,m}+2+\sigma}{2} \right)}{\Gamma\left(\frac{\lambda_{k,a,m}+2-\sigma}{2} \right)} \frac{K_{(\lambda_{k,a,m}+1+\sigma)/2}\left((\delta + \frac{2}{a} r^a)/2 \right)}{K_{(\lambda_{k,a,m}+1-\sigma)/2}\left((\delta + \frac{2}{a} r^a)/2 \right)} \left(\delta + \tfrac{2}{a} r^a \right)^{-\sigma}
\end{aligned}
$$

$$\geq \left(\frac{a}{2}\right)^{\sigma} \delta^{\sigma} \frac{\Gamma\left(\frac{\lambda_a+2+\sigma}{2}\right)}{\Gamma\left(\frac{\lambda_a+2-\sigma}{2}\right)} \left(\delta + \tfrac{2}{a} r^a\right)^{-\sigma} = \left(\frac{a}{2}\right)^{\sigma} B^{\delta}_{\lambda_a,\sigma} \left(\delta + \tfrac{2}{a} r^a\right)^{-\sigma}.$$

Therefore,

$$\begin{aligned}
\left\langle (-\triangle_{k,a})_{\sigma} f, f \right\rangle_{L^2\left(\mathbb{R}^N, \vartheta_{k,a}(x)dx\right)} &\geq \sum_{m,j} \left(\frac{2a}{\delta}\right)^{\sigma} \left(B^{\delta}_{\lambda_{k,a,m},\sigma}\right)^2 \int_0^{\infty} |f_{m,j}(r)|^2 \frac{\omega^{\delta,a}_{\lambda_{k,a,m},\sigma}(r)}{\omega^{\delta,a}_{\lambda_{k,a,m},-\sigma}(r)} d\mu_{a,\lambda_a}(r) \\
&\geq \left(\frac{a}{2}\right)^{\sigma} B^{\delta}_{\lambda_a,\sigma} \sum_{m,j} \int_0^{\infty} |f_{m,j}(r)|^2 \left(\delta + \tfrac{2}{a} r^a\right)^{-\sigma} d\mu_{a,\lambda_a}(r) \\
&= \left(\frac{a}{2}\right)^{\sigma} B^{\delta}_{\lambda_a,\sigma} \int_{\mathbb{R}^N} \frac{|f(x)|^2}{\left(\delta + \frac{2}{a} \|x\|^a\right)^{\sigma}} \vartheta_{k,a}(x)\, dx.
\end{aligned}$$

The proof of Theorem 3.1 is completed.

3.4 Relationship with $\mathfrak{sl}_2$-representation

Consider the map

$$\alpha^{(m)}_{k,a} : \mathcal{H}^m_k(\mathbb{R}^N)|_{S^{N-1}} \otimes L^2(\mathbb{R}_+, r^{2\langle k\rangle+N+a-3}dr) \longrightarrow L^2\big(\mathbb{R}^N, \vartheta_{k,a}(x)dx\big)$$

defined by

$$\alpha^{(m)}_{k,a}(p \otimes f)(x) = p\big(\frac{x}{\|x\|}\big) f(\|x\|)$$

for $p \in \mathcal{H}^m_k(\mathbb{R}^N)|_{S^{N-1}}$ and $f \in L^2(\mathbb{R}_+, r^{2\langle k\rangle+N+a-3}dr)$.

It follows that the unitary representation $\Omega_{k,a}$ of $\widetilde{SL(2,\mathbb{R})}$ on the Hilbert space $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$ induces a family of unitary operators $\Omega^{(m)}_{k,a}(\gamma_z)$ on $L^2(\mathbb{R}_+, r^{2\langle k\rangle+N+a-3}dr)$ such that (see [BSKØ, (4.3)])

$$\alpha^{(m)}_{k,a}\Big(p \otimes \Omega^{(m)}_{k,a}(\gamma_z)(f)\Big) = \Omega_{k,a}(\gamma_z)\Big(\alpha^{(m)}_{k,a}(p \otimes f)\Big). \tag{3.12}$$

Then from (2.7), (2.8), $\pi_K(\lambda_{k,a,m})$ is integrable and $d\Omega^{(m)}_{k,a} = \pi_K(\lambda_{k,a,m})$. And in [8, Section 4.1] they showed that the unitary operator $\Omega^{(m)}_{k,a}(\gamma_z)$ on $L^2\left(\mathbb{R}_+, r^{\,2\langle k\rangle+N+a-3}dr\right)$ can be expressed as

$$\Omega^{(m)}_{k,a}(\gamma_z) f(r) = \int_0^{\infty} \Lambda^{(m)}_{k,a}(r,s;z) f(s) s^{2\langle k\rangle+N+a-3} ds, \tag{3.13}$$

where $\Lambda^{(m)}_{k,a}(r,s;z)$ has its closed formula (see [8, (4.11)])

$$\Lambda^{(m)}_{k,a}(r,s;z) = \frac{(rs)^{-\langle k\rangle-\frac{N}{2}+1}}{\sinh\ z} e^{-\frac{\coth z}{a}(r^a+s^a)} I_{\lambda_{k,a,m}}\left(\frac{\frac{2}{a} r^{\frac{a}{2}} s^{\frac{a}{2}}}{\sinh z}\right).$$

The integral on the right hand side of (3.13) converges for $f \in L^2\left(\mathbb{R}_+, r^{\,2\langle k\rangle+N+a-3}dr\right)$ if $\Re z > 0$ and for all f in the dense subspace of $L^2\left(\mathbb{R}_+, r^{\,2\langle k\rangle+N+a-3}dr\right)$ spanned by the functions $\left\{\psi^{(a)}_{l,m}(r) : \, l \in \mathbb{N}\right\}$ if $\Re z = 0$ (see (2.3) for the definition of $\psi^{(a)}_{l,m}(r)$). We give an explicit expression of $\Omega^{(m)}_{k,a}(\gamma_z)$ in this section via the a-deformed Laguerre operator $L_{a,\alpha}$ (see [4] for the case of $a = 2$ on such expression).

Theorem 3.12. *([41]) Assume $\lambda_{k,a,m} \geq -1/2$, $\Re z \geq 0$ and $s > 0$. Then $\Omega^{(m)}_{k,a}(\gamma_z)$ acting on $L^2\left(\mathbb{R}_+, r^{\,2\langle k\rangle+N+a-3}dr\right)$ has the form*

$$\Omega^{(m)}_{k,a}(\gamma_z)\, f(s) = s^m I_{a,\lambda_{k,a,m};z}\left((\cdot)^{-m} f\right)(s)\,.$$

Thus

$$\left.\frac{d}{dz}\right|_{z=0} \Omega^{(m)}_{k,a}(\gamma_z)\, f(s) = \pi_K(\lambda_{k,a,m})(\mathbf{k})f(s) = -s^m \frac{1}{a} L_{a,\lambda_{k,a,m}}\left((\cdot)^{-m} f\right)(s)\,.$$

Proof. We can take α as $\lambda_{k,a,m}$ in Lemma 3.8, then we get

$$\mathcal{T}^{a,\lambda_{k,a,m}}_r q_{a,\lambda_{k,a,m};z}(s) = (rs)^{-m}\Lambda^{(m)}_{k,a}(r,s;z)\,. \tag{3.14}$$

For every f in the dense subspace of $L^2\left(\mathbb{R}_+, r^{\,2\langle k\rangle+N+a-3}dr\right)$ spanned by the functions $\left\{\psi^{(a)}_{l,m}(r) : \, l \in \mathbb{N}\right\}$, we have

$$\begin{aligned}
\Omega^{(m)}_{k,a}(\gamma_z)\, f(s) &= \int_0^\infty f(r)\Lambda^{(m)}_{k,a}(r,s;z)\, r^{\,2\langle k\rangle+N+a-3}dr \\
&= s^m \int_0^\infty r^{-m} f(r)\mathcal{T}^{a,\lambda_{k,a,m}}_r q_{a,\lambda_{k,a,m};z}(s) r^{\,2m+2\langle k\rangle+N+a-3}dr \\
&= s^m I_{a,\lambda_{k,a,m};z}\left((\cdot)^{-m} f\right)(s)\,.
\end{aligned}$$

□

Remark 3.13. *i). From this theorem, the spherical harmonic expansion of the (k,a)-generalized Laguerre semigroup* (3.8) *can be derived directly. Taking p as Y^m_i in* (3.12)*, we have*

$$\Omega_{k,a}(\gamma_z)\left(f_{m,i}(r)Y^m_i(x')\right) = Y^m_i(x')\Omega^{(m)}_{k,a}(\gamma_z)\left(f_{m,i}(r)\right)$$

for every $f_{m,i}(r)Y^m_i(x')$, Then by summing up, we can derive (3.8)*.*
ii). Taking $m = 0$, we get the formula of $\triangle_{k,a}$ on radial Schwartz functions $f = f_0(\|\cdot\|)$, $f_0 \in \mathcal{S}(\mathbb{R}_+)$,

$$\triangle_{k,a} f(x) = -L_{a,\lambda_a}(f_0)(r)\,,\ r = \|x\|\,.$$

This is equivalent to the formula of Dunkl Laplacian $\triangle_k$ on radial functions in [30, Proposition 4.15], i.e.,

$$\triangle_k = \frac{d^2}{dr^2} + \frac{2\langle k\rangle + N - 1}{r}\frac{d}{dr}.$$

iii). In [4], there is a comparable result on expressing each $\mathfrak{m}$-component of the representation $\left(\omega_{k,a}, W_{k,a}\left(\mathbb{R}^N\right)\right)$ via differential operators.

4 The generalized translation operator

Assume $2\langle k\rangle + N + a - 3 \geq 0$. For $a = \frac{2}{n}$, $n \in \mathbb{N}$, one can define the (k,a)-generalized translation on $L^2\left(\mathbb{R}^N, \vartheta_{k,a}(x)\,dx\right)$ as

$$F_{k,a}(\tau_y f)(\xi) := B_{k,a}(y,\xi)\, F_{k,a}(f)(\xi), \quad \xi \in \mathbb{R}^N.$$

The above definition makes sense because for $a = \frac{2}{n}$, $n \in \mathbb{N}$, $F_{k,a}$ is an isometry on $L^2\left(\mathbb{R}^N, \vartheta_{k,a}(x)\,dx\right)$ from the inversion formulae, and its integral kernel $B_{k,a}(x,y)$ satisfies the uniform boundedness condition (2.14). In this case the (k,a)-generalized translation can also be written via an integral as

$$\tau_y f(x) = c_{k,a} \int_{\mathbb{R}^N} B_{k,a}(x,\xi)\, B_{k,a}(y,\xi)\, F_{k,a}(f)(\xi)\, \vartheta_{k,a}(\xi)\, d\xi$$

for $f \in \mathcal{L}_k^1\left(\mathbb{R}^N\right)$, where

$$\mathcal{L}_k^1\left(\mathbb{R}^N\right) := \left\{ f \in L^1\left(\mathbb{R}^N, \vartheta_{k,a}(x)\,dx\right) :\ F_{k,a}(f) \in L^1\left(\mathbb{R}^N, \vartheta_{k,a}(x)\,dx\right)\right\}.$$

This formula holds true on Schwartz space $\mathcal{S}\left(\mathbb{R}^N\right)$ since $\mathcal{S}\left(\mathbb{R}^N\right)$ is a subspace of $\mathcal{L}_k^1\left(\mathbb{R}^N\right)$.

For the two cases of $a = 1$ and $a = 2$, the analytic structure is richer because we have the formulas for radial functions of the generalized translation for the two special cases. The radial formula for $a = 2$ (for the Dunkl translation) was found by Rösler [34] and for $a = 1$ it was found by S. Ben Saïd and L. Deleaval [5]. The generalized translation operator τ_x corresponds to the classical translation operator $f \mapsto f(x - \cdot)$ for $a = 1$, and corresponds to $f \mapsto f(x + \cdot)$ for $a = 2$. This is because for $a = 1$, the inversion formula of the generalized Fourier analysis is $F_{k,1}^{-1}(f) = F_{k,1}(f)$, and for $a = 2$, the inversion formula is $\left(F_k^{-1} f\right)(x) = \left(F_k f\right)(-x)$. We will study the generalized translation for the two cases in the following. And in particular, we will investigate the support of the generalized translations of radial functions. Such results in the chapter for $a = 2$ was in my paper [39] and that for $a = 1$ was in my paper [40]. And for the case of $a = 1$, we will need to study the metric space (also contained in [40]) corresponding to the $(k,1)$-generalized analysis in order to investigate the support of translations of functions.

4.1 The case of $a = 2$ (the Dunkl case)

The generalized translation for $a = 2$ (called Dunkl translation) on $L^1(m_k)$ can also be defined by the intertwining operator as

$$\tau_x f(y) = (V_k)_y (V_k)_x \left[(V_k)^{-1}(f)(x+y) \right].$$

Here are some basic properties of Dunkl translations.

1. $(identity)\ \tau_0 = I$;
2. $(Symmetry)\ \tau_x f(y) = \tau_y f(x),\ x,\ y \in \mathbb{R}^N,\ f \in \mathcal{S}(\mathbb{R}^N)$;
3. $(Commutativity)\ T_\xi(\tau_x f) = \tau_x(T_\xi f),\ x,\ \xi \in \mathbb{R}^N$;
4. $(Skew-symmetry)$
$$\int_{\mathbb{R}^N} \tau_x f(y) g(y) dm_k(y) = \int_{\mathbb{R}^N} f(y)\tau_{-x} g(y) dm_k(y),\ x \in \mathbb{R}^N,\ f,\ g \in \mathcal{S}(\mathbb{R}^N).$$

The Dunkl translations can be defined on $L^p(m_k),\ 1 \le p \le \infty$ in the distributional sense due to the latter formula. Further,

$$\int_{\mathbb{R}^N} \tau_x f(y) dm_k(y) = \int_{\mathbb{R}^N} f(y) dm_k(y),\ x \in \mathbb{R}^N,\ f \in \mathcal{S}(\mathbb{R}^N). \tag{4.1}$$

The following formula for radial functions was first proved by Rösler [34] for Schwartz functions, and was then extended to all continuous radial functions in [15]:

$$\tau_x f(-y) = \int_{\mathbb{R}^N} (\widetilde{f} \circ A)(x,\ y,\ \eta) d\mu_x(\eta),\ x,\ y \in \mathbb{R}^N, \tag{4.2}$$

where $f(x) = \widetilde{f}(\|x\|)$ and

$$A(x,\ y,\ \eta) = \sqrt{\|x\|^2 + \|y\|^2 - 2\langle y,\ \eta\rangle} = \sqrt{\|x\|^2 - \|\eta\|^2 + \|y - \eta\|^2}.$$

For any $\eta \in co(G.x)$, we have

$$A(x,\ y,\ \eta) \ge \min_{g \in G} \|y - gx\|. \tag{4.3}$$

It follows from the symmetry of Dunkl translations that (see [21])

$$\tau_{-x} f(y) = \tau_y f(-x) = \tau_x f(-y),\ x,\ y \in \mathbb{R}^N,\ f \in \mathcal{S}_{rad}(\mathbb{R}^N).$$

The Dunkl convolution of Schwartz functions is defined by

$$(f * g)(x) = \int_{\mathbb{R}^N} f(y)\tau_x g(-y) dm_k(y),$$

or can be written as

$$(f * g)(x) = \int_{\mathbb{R}^N} (F_k f)(\xi)(F_k g)(\xi) E(ix,\ \xi) dm_k(\xi).$$

The following are some basic properties of Dunkl convolution,

1. $F_k(f * g) = F_k f \cdot F_k g$;

2. $F_k(f\cdot g)=F_kf*F_kg$;

3. $f*g=g*f$;

4. $(f*g)*h=f*(g*h)$;

5. $\|f*g\|_{2,\,k}\leq\|f\|_{1,\,k}\|g\|_{2,\,k},\ f\in L^1(m_k),\ g\in L^2(m_k)$.

The following theorem (part ii) shows that the support of $\tau_{-x}f$ obtained in [14, Theorem 1.7] (part i of the following Theorem) is precise when the multiplicity function $k>0$. The preciseness has been proved for characteristic functions by Gallardo and Rejeb [21] and we extend the result to any nonnegative radial functions on $L^2(m_k)$ in the following theorem. Here $B(x,r)$ denotes the closed ball $\{y\in\mathbb{R}^N:\|x-y\|\leq r\}$.

Theorem 4.1. *If $f\in L^2(m_k)$ and $suppf\subseteq B(0,r)$, then for any $x\in\mathbb{R}^N$*
i).([14, Theorem 1.7])

$$supp\tau_xf(-\cdot)\subseteq\bigcup_{g\in G}B(gx,\ r).$$

ii).([39]) If the multiplicity function $k>0$ and let f be a nonnegative radial function on $L^2(m_k)$, $suppf=B(0,r)$, then

$$supp\tau_xf(-\cdot)=\bigcup_{g\in G}B(gx,\ r).$$

Proof. ii). It suffices to prove that

$$supp\tau_xf(-\cdot)\supseteq\bigcup_{g\in G}B(gx,\ r).$$

Firstly, we will prove for continuous nonnegative radial functions. Suppose there exists a $y\in\bigcup_{g\in G}B(gx,\ r)$, that is, there exists a $g\in G$, $\|y-gx\|\leq r$, such that $y\notin supp\tau_xf\,(-\cdot)$, that is, there exists $\varepsilon>0$, for any $z\in B(y,\varepsilon)$,

$$0=\tau_xf(-z)=\int_{\mathbb{R}^N}\widetilde{f}(\sqrt{\|x\|^2+\|z\|^2-2\langle z,\ \eta\rangle})d\mu_x(\eta),$$

then

$$\widetilde{f}(\sqrt{\|x\|^2+\|z\|^2-2\langle z,\ \eta\rangle})=0,\ \ for\ any\ \eta\in supp\mu_x.$$

By a result of Gallardo and Rejeb (see [21]), that the orbit of x, $G.x$, is contained in the support of μ_x if $k>0$, for the above g we can select $\eta=gx$, then $f(z-gx)=\widetilde{f}(\|z-gx\|)=0$.

For any $z_1 \in B(y - gx, \varepsilon)$, $z_1 + gx \in B(y, \varepsilon)$, and so $f(z_1) = f(z_1 + gx - gx) = 0$, which means $y - gx \notin suppf$, and this leads to a contradiction to that $suppf = B(0, r)$.

Then for any nonnegative radial functions f on $L^2(m_k)$, $suppf = B(0, r)$, by the density of continuous functions with compact support $B(0, r)$ in $L^2(B(0, r), m_k)$, there exists a sequence of continuous nonnegative radial functions g_n whose support is $B(0, r)$, such that $f/2$ can be approximated by g_n with respect to L^2-norm. So for any nonnegative smooth function φ on $\mathbb{R}^N$ with compact support, $\int g_n \varphi \to \int \frac{f}{2} \varphi$. If $(supp\varphi)^\circ \cap B(0, r) \neq \varnothing$, then $\int f \varphi > 0$, where A° stands for the interior of A for any $A \subseteq \mathbb{R}^N$. So there exists a sufficiently large natural number L such that $\int g_L \varphi < \int f \varphi$. If $(supp\varphi)^\circ \cap B(0, r) = \varnothing$, then for any $n \in \mathbb{N}$, $\int g_n \varphi = \int f \varphi = 0$. So for any nonnegative smooth function φ on $\mathbb{R}^N$ with compact support, $\int g_L \varphi \leq \int f \varphi$. Thus $g_L \leq f$ *a.e.* and $\int \tau_{-x} g_L \cdot \varphi \leq \int \tau_{-x} f \cdot \varphi$ by positivity of Dunkl translations on radial functions. Let $D = (supp\tau_{-x} f)^c$, then D is the largest open set such that $0 = \int \tau_{-x} f \cdot \varphi$ for any smooth functions functions φ with compact support in D. If $\varphi \geq 0$, then $\int \tau_{-x} g_L \cdot \varphi = 0$. Then by $\tau_{-x} g_L \geq 0$,

$$\bigcup_{g \in G} B(gx,\ r) = supp\tau_{-x} g_L \subseteq D^c = supp\tau_{-x} f. \qquad \square$$

Remark 4.2. *This theorem does not hold for $k \geqslant 0$. For example, for any nontrivial finite reflection group G, we can take $k = 0$. Then $supp\tau_x f(-\cdot) = B(x, r)$ when $suppf = B(0, r)$ and is obviously not $\bigcup_{g \in G} B(gx,\ r)$ since G is nontrivial. We refer to [21, Example 3.1] for more counterexamples.*

4.2 The case of $a = 1$

The generalized translation τ_y for $a = 1$ satisfies the following properties:

(1). For every $x, y \in \mathbb{R}^N$,

$$\tau_y f(x) = \tau_x f(y), \ f \in \mathcal{S}\left(\mathbb{R}^N\right). \tag{4.4}$$

(2). For every $y \in \mathbb{R}^N$,

$$\int_{\mathbb{R}^N} \tau_y f(x)\, g(x)\, \vartheta_{k,1}(x)\, dx = \int_{\mathbb{R}^N} f(x)\, \tau_y g(x)\, \vartheta_{k,1}(x)\, dx, \quad f, g \in \mathcal{S}\left(\mathbb{R}^N\right). \tag{4.5}$$

Here the property (1) corresponds to $\tau_y f(x) = \tau_{-x} f(-y)$ and (2) corresponds to the skew-symmetry in classical Fourier analysis and Dunkl analysis.

For any radial function $f \in \mathcal{S}(\mathbb{R}^N)$, i.e., $f(x) = f_0(\|x\|)$, $\langle k\rangle + \frac{N-2}{2} > 0$, τ_y can be expressed as follows (see [5])

$$\tau_y f(x) = \frac{\Gamma\left(\frac{N-1}{2} + \langle k\rangle\right)}{\sqrt{\pi}\Gamma\left(\frac{N-2}{2} + \langle k\rangle\right)} \times V_k\left(\int_{-1}^{1} f_0\left(\|x\| + \|y\| - \sqrt{2(\|x\|\|y\| + \langle \cdot, y\rangle)}u\right)\left(1-u^2\right)^{\frac{N}{2}+\langle k\rangle - 2} du\right)(x). \quad (4.6)$$

And so τ_y is positive on radial functions and can be extended as a bounded operator to the space of all radial functions on $L^p\left(\mathbb{R}^N, \vartheta_{k,1}(x)\,dx\right)$, $1 \le p \le 2$. Further, if f is a nonnegative radial function on $L^1\left(\mathbb{R}^N, \vartheta_{k,1}(x)\,dx\right)$, then

$$\int_{\mathbb{R}^N} \tau_y f(x)\,\vartheta_{k,1}(x)\,dx = \int_{\mathbb{R}^N} f(x)\,\vartheta_{k,1}(x)\,dx. \quad (4.7)$$

The authors in [5] also gave a special case of the formula for radial functions

$$\tau_y\left(e^{-\lambda\|\cdot\|}\right)(x) = \Gamma\left(\langle k\rangle + \frac{N-1}{2}\right) e^{-\lambda(\|x\|+\|y\|)} V_k\left(\widetilde{I}_{\langle k\rangle + \frac{N-3}{2}}\left(\lambda\sqrt{2(\|x\|\|y\| + \langle x, \cdot\rangle)}\right)\right)(y). \quad (4.8)$$

Now we construct the metric space corresponding to the $(k,1)$-generalized setting. For $x, y \in \mathbb{R}^N$, in view of the expression (4.6) of $(k,1)$-generalized translation operators, we define a function d from $\mathbb{R}^N \times \mathbb{R}^N$ to $\mathbb{R}$ as

$$\begin{aligned} d(x,y) :&= \sqrt{\|x\| + \|y\| - \sqrt{2(\|x\|\|y\| + \langle x, y\rangle)}} \\ &= \sqrt{\|x\| + \|y\| - 2\sqrt{\|x\|\|y\|}\cos\frac{\theta}{2}} \ge \left|\sqrt{\|x\|} - \sqrt{\|y\|}\right|, \end{aligned}$$

where $\theta = \arccos \frac{\langle x,y\rangle}{\|x\|\|y\|}$, $0 \le \theta \le \pi$.

Proposition 4.3. *([40]) The function $d(x,y)$ is a metric.*

Proof. The symmetry property is obvious. For the positivity property, if $d(x,y) = 0$, then $\|x\| = \|y\|$ and $\sqrt{\|x\| - \|x\|\cos\frac{\theta}{2}} = 0$ leading to $\theta = 0$. Hence $x = y$.

Then we turn to prove the triangle inequality. Let

$$\alpha = \arccos\frac{\langle x, y\rangle}{\|x\|\|y\|},\ \beta = \arccos\frac{\langle x, z\rangle}{\|x\|\|z\|},\ \gamma = \arccos\frac{\langle z, y\rangle}{\|z\|\|y\|},\ 0 \le \alpha,\ \beta,\ \gamma \le \pi.$$

Then we have $\beta + \gamma \ge \alpha$ from the triangle inequality of the spherical distance. Therefore,

$$d(x,y) \le \sqrt{\|x\| + \|y\| - 2\sqrt{\|x\|\|y\|}\cos\frac{\beta+\gamma}{2}}.$$

It suffices to show that

$$\sqrt{\|x\| + \|y\| - 2\sqrt{\|x\|\,\|y\|}\cos\frac{\beta+\gamma}{2}} \le d\,(x,z) + d\,(z,y)\,.$$

Take the square of the above inequality and eliminate some items. It suffices to show the following inequality,

$$\begin{aligned}\|x\|\,\|y\|\sin^2\frac{\beta+\gamma}{2} &+ \|x\|\,\|z\|\sin^2\frac{\beta}{2} + \|z\|\,\|y\|\sin^2\frac{\gamma}{2} + 2\,\|z\|\,\sqrt{\|x\|\,\|y\|}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}\\ &+ 2\,\|x\|\,\sqrt{\|z\|\,\|y\|}\cos\frac{\beta}{2}\cos\frac{\beta+\gamma}{2} + 2\,\|y\|\,\sqrt{\|z\|\,\|x\|}\cos\frac{\gamma}{2}\cos\frac{\beta+\gamma}{2}\\ &\ge 2\,\|x\|\,\sqrt{\|z\|\,\|y\|}\cos\frac{\gamma}{2} + 2\,\|y\|\,\sqrt{\|z\|\,\|x\|}\cos\frac{\beta}{2} + 2\,\|z\|\,\sqrt{\|x\|\,\|y\|}\cos\frac{\beta+\gamma}{2}.\end{aligned}$$

And the inequality is equivalent to

$$\left(\sqrt{\|x\|\,\|y\|}\sin\frac{\beta+\gamma}{2} - \sqrt{\|x\|\,\|z\|}\sin\frac{\beta}{2} - \sqrt{\|z\|\,\|y\|}\sin\frac{\gamma}{2}\right)^2 \ge 0.$$

Proposition 4.3 is therefore proved. □

Remark 4.4. *i). For the one dimensional case, the metric $d(x,y)$ recedes to*

$$d(x,y) = \begin{cases} \sqrt{|x-y|}, & xy \le 0 \\ \left|\sqrt{|x|} - \sqrt{|y|}\right|, & xy > 0 \end{cases}.$$

The ball with respect to this metric in this case was already used in [6] to define the generalized Hardy–Littlewood maximal operator.

ii). A continuous rectifiable curve between two distinct points does not necessarily exist with respect to this metric. For example, if we take $x = -1$ and $y = 1$ for the one dimensional case, then distance between x and y with respect to the induced length metric is no less than $\sup\limits_n \sum_{i=1}^n \sqrt{\frac{2}{n}} = \infty$.

Proposition 4.5. *([40]) $\left(\mathbb{R}^N, d\right)$ is a complete metric space.*

Proof. We will show that $d(x,y)$ is equivalent to the Euclidean metric. If $y_n \to y$ with respect to the Euclidean metric, then $d\,(y_n, y) \to 0$ obviously. If $d\,(y_n, y) \to 0$, then $\|y_n\| \to \|y\|$. Denote $\theta_n = \arccos\frac{\langle y_n, y\rangle}{\|y_n\|\|y\|}$. Then

$$\sqrt{2\,\|y\| - 2\,\|y\|\lim_{n\to\infty}\cos\frac{\theta_n}{2}} = 0.$$

So, $\lim_{n\to\infty}\cos\theta_n = 1$ and $\lim_{n\to\infty}\langle y_n, y\rangle = \|y\|^2$. Hence

$$\lim_{n\to\infty}\|y_n - y\| = \lim_{n\to\infty}\sqrt{\|y_n\|^2 + \|y\|^2 - 2\,\langle y_n, y\rangle} = 0.$$

□

The closure of an open ball in a metric space is not necessarily the closed ball. In [45] the authors gave a sufficient but not necessary condition such that the closure of the open ball is the closed ball. They showed that if the metric is weakly convex, i.e., for any two different points x and y, there exists $z \neq x, y$, such that $d(x,y) = d(x,z) + d(z,y)$. The metric d we are concerned with is not weakly convex obviously but the closure of the open ball with respect to this metric is still the closed ball.

Theorem 4.6. *([40]) The closure* $\overline{B_0(x,r)}$ *of the open ball* $B_0(x,r) = \{y : d(y,x) < r\}$, $r > 0$ *is the closed ball* $B(x,r)$.

Proof. Let y be a point $\mathbb{R}^N$ distinct from x such that $d(x,y) = r$. We show that for any $\varepsilon > 0$, there exists $z \in B(y,\varepsilon)$, such that $d(x,z) < r = d(x,y)$. Let M_x, $x \neq 0$ be the mapping $M_x : \mathbb{R}^N \to [0,+\infty)$, $y \mapsto d(x,y)$ and $L_x(y) := M_x(y)^2$. It suffices to show that the function L_x takes no minimum point on $\mathbb{R}^N$ except at $y = x$. Notice that L_x is differentiable on $\mathbb{R}^N \setminus \{0\}$. We calculate the points such that

$$0 = \frac{\partial L_x}{\partial y_i} = \frac{y_i}{\|y\|} - \frac{\|x\| \frac{y_i}{\|y\|} + x_i}{\sqrt{2\left(\|x\|\,\|y\| + \langle x,y\rangle\right)}}, \quad i = 1, 2, \cdots, N.$$

By summing up the square, we get

$$\sqrt{2\left(\|x\|\,\|y\| + \langle x,y\rangle\right)} = 2\,\|y\| \text{ and } x_i = t y_i, \text{ where } t = 2 - \frac{\|x\|}{\|y\|}.$$

Thus $2y_i = |t|\, y_i + t y_i$ and $y = x$. For the point $y = 0$, consider the function

$$\begin{aligned} L(y_1) := L_x(y_1, 0, \ldots, 0) &= \|x\| + |y_1| - \sqrt{2\left(\|x\|\,|y_1| + x_1 y_1\right)} \\ &= \begin{cases} \|x\| + y_1 - \sqrt{2\left(\|x\| + x_1\right) y_1}, & y_1 \geq 0 \\ \|x\| - y_1 - \sqrt{-2\left(\|x\| - x_1\right) y_1}, & y_1 < 0. \end{cases} \end{aligned}$$

It does not take minimum at $y_1 = 0$ obviously. Therefore, L_x takes no minimum point on $\mathbb{R}^N$ except at $y = x$. □

The metric space $\left(\mathbb{R}^N, d\right)$, rather than the standard Euclidean metric space, is the natural metric space corresponding to the $(k,1)$-generalized setting when metric is involved due to the expression of the $(k,1)$-generalized translation operators. In the following theorem we give a characterization of support of the $(k,1)$-generalized translation of a function supported in $B(0,r) = \left\{y \in \mathbb{R}^N : \sqrt{\|y\|} \leq r\right\}$.

Theorem 4.7. *([40]) Let $f = f_0(\|\cdot\|)$ be a nonnegative radial function on $L^2\left(\mathbb{R}^N, \vartheta_{k,1}(x)\,dx\right)$, $\operatorname{supp} f = B(0,r)$, then*

$$\operatorname{supp}\tau_x f = \bigcup_{g\in G} B(gx, r).$$

Proof. We extend the formula of $(k,1)$-generalized translations on radial Schwartz functions (4.6) to all continuous radial functions on $L^2\left(\mathbb{R}^N, \vartheta_{k,1}(x)\,dx\right)$ first. The proof goes similar to the Lemma 3.4 in [15]. The only difference is to take the set A_n in the proof as

$$A_n \equiv A_n(y) := \left\{x \in \mathbb{R}^N : 2^{-n} \leq \left|\sqrt{\|x\|} - \sqrt{\|y\|}\right| \leq \sqrt{\|x\|} + \sqrt{\|y\|} \leq 2^n\right\}$$

for $n \in \mathbb{N}$ and $n \geq \frac{1}{2}\left[\log\ \|y\| / \log\ 2\right] + 1$, since

$$\left|\sqrt{\|x\|} - \sqrt{\|y\|}\right| \leq \sqrt{\|x\| + \|y\| - \sqrt{2\left(\|x\|\,\|y\| + \langle\eta, y\rangle\right)}u} \leq \sqrt{\|x\|} + \sqrt{\|y\|}$$

for $\eta \in co(G.x)$ and $u \in [-1,1]$.

Then we prove the theorem for continuous nonnegative radial functions. For the proof of $\mathit{supp}\tau_x f \subseteq \bigcup_{g\in G} B(gx,r)$, from the radial formula (4.6) of $(k,1)$-generalized translations and notice that for any $\eta \in co(G.x)$ and $u \in [-1,1]$,

$$\sqrt{\|x\| + \|y\| - \sqrt{2\left(\|x\|\,\|y\| + \langle\eta, y\rangle\right)}u} \geq \min_{g\in G} d\left(gx, y\right), \tag{4.9}$$

we have $\tau_x f(y) = 0$ for $y \in \left(\bigcup_{g\in G} B(gx,r)\right)^c$ if $\operatorname{supp} f \subseteq B(0,r)$. For the converse part $\bigcup_{g\in G} B(gx,r) \subseteq \operatorname{supp}\tau_x f$, we will show that $\bigcup_{g\in G} B_0(gx,r) \subseteq \operatorname{supp}\tau_x f$ first. Suppose there exists a $y \in \bigcup_{g\in G} B_0(gx,r)$ for which $y \notin \operatorname{supp}\tau_x f$. Then there exists $\varepsilon > 0$, such that for any $z \in B(y,\varepsilon)$, we have $z \in \bigcup_{g\in G} B_0(gx,r)$ (that is, there also exists a $g \in G$ such that $d(z, gx) < r$) and

$$\begin{aligned} 0 = \tau_x f(z) =& \frac{\Gamma\left(\frac{N-1}{2} + \langle k\rangle\right)}{\sqrt{\pi}\Gamma\left(\frac{N-2}{2} + \langle k\rangle\right)} \times \\ & \int_{\mathbb{R}^N}\int_{-1}^{1} f_0\left(\|x\| + \|z\| - \sqrt{2\left(\|x\|\,\|z\| + \langle\eta, z\rangle\right)}u\right)\left(1-u^2\right)^{\frac{N}{2}+\langle k\rangle-2} du d\mu_x(\eta). \end{aligned}$$

Thus

$$f_0\left(\|x\| + \|z\| - \sqrt{2\left(\|x\|\,\|z\| + \langle\eta, z\rangle\right)}u\right) = 0$$

for any $\eta \in \operatorname{supp}\mu_x$ and $u \in [-1,1]$. Then from a result of Gallardo and Rejeb (see [21]), that the orbit of x, $G.x$, is contained in $\operatorname{supp}\mu_x$, we can select $u = 1$ and $\eta = gx$ for the above

g. Then we get $f_0\left(d\left(gx, z\right)^2\right) = 0$ for all $z \in B(y, \varepsilon)$. But $d\left(z, gx\right) < r$, which contradicts to that $\operatorname{supp} f_0 = [0, r^2]$. Then from Theorem 3.4, we get $\bigcup_{g \in G} B(gx, r) \subseteq \operatorname{supp} \tau_x f$.

The conclusion for all nonnegative radial function on $L^2\left(\mathbb{R}^N, \vartheta_{k,1}\left(x\right) dx\right)$ can then be derived from the density of continuous functions with compact support $B(0, r)$ in $L^2(B(0, r), \vartheta_{k,1}\left(x\right) dx)$ and the positivity of the $(k, 1)$-generalized translations on radial functions as in the proof of Theorem 4.1. $\square$

5 Calderón–Zygmund theory

We recall the Calderón–Zygmund theory on general homogeneous space first. Let (X, d) be a metric space. Denote $B(x,r)$ to be the ball $B\left(x,r\right) := \{y \in X : d\left(x,y\right) \le r\}$ for $x \in X$. If there exists a doubling measure m, i.e., there exists a measure m such that for some absolute constant C,

$$m\left(B\left(x,2r\right)\right) \le Cm\left(B\left(x,r\right)\right),\ \forall x \in \mathbb{R}^N,\ r > 0, \tag{5.1}$$

then (X, d) is a space of homogeneous type. The Calderón–Zygmund theory on a space of homogeneous type (X, d, m) says that for $f \in L^1(X,m) \cap L^2(X,m)$ and $\lambda > \frac{\|f\|_1}{m(X)}$, there exists the Calderón–Zygmund decomposition $f = h + b$ with $b = \sum_j b_j$ and a sequence of balls $(B(y_j, r_j))_j = (B_j)_j$ such that for some absolute constant C,

(i) $\|h\|_\infty \le C\lambda$;

(ii) $supp(b_j) \subset B_j$;

(iii) $\displaystyle\int_{B_j} b_j(x)dm(x) = 0$;

(iv) $\|b_j\|_{L^1(X,m)} \le C\,\lambda\, m(B_j)$;

(v) $\displaystyle\sum_j m(B_j) \le C\,\frac{\|f\|_{L^1(X,m)}}{\lambda}$.

From the Calderón–Zygmund decomposition one can deduce that for a bounded operator S on $L^2(X,m)$ associated with kernel $K(x,y)$, if $K(x,y)$ satisfies the Hörmander type condition

$$\int_{d(x,y)>2d(y,y_0)} |K(x,\ y) - K(x,\ y_0)|\, dm(x) \le C,\quad y, y_0 \in \mathbb{R}^N,$$

then the operator S can be extended to a bounded operator on $L^p(X,m)$ $(1 < p \le 2)$ and a weakly bounded operator on $L^1(X,m)$, i.e.,

$$\|Sf\|_p \le C_p\|f\|_p,\quad \text{if } 1 \le p \le 2,$$

and

$$m\left\{x :\ |S(f)(x)| > \lambda\right\} \le C_1\frac{\|f\|_p}{\lambda}.$$

We refer to [13, Chapter III] for this theory.

We adapt the Calderón–Zygmund theory on homogeneous space to the context of (k,a)-generalized Fourier analysis for $a=2$ (Dunkl setting) and $a=1$, respectively. For $a=2$, such adaption was already given in [2], and for $a=1$, the adaption was contained in my paper [40].

We define the distance between the two orbits $G.x$ and $G.y$ as $d_G(x,y)=\min_{g\in G} d(gx,y)$, where $d(x,y)$ denotes the metric corresponding to the (k,a)-generalized Fourier analysis for $a=2$ and 1. For $a=2$, $d(x,y)$ denotes the Euclidean metric, i.e., $d(x,y):=\|x-y\|$, and for $a=1$, $d(x,y):=\sqrt{\|x\|+\|y\|-\sqrt{2(\|x\|\,\|y\|+\langle x,y\rangle)}}$, as was studied in the last chapter.

Denote $dm_{k,a}(x)=\vartheta_{k,a}(x)\,dx$. We then show that the metric spaces $(X,d,m_{k,a})$ for both $a=2$ and $a=1$ are of homogeneous type. For $a=2$, it was already shown in Section 2.2 that the measure is doubling with respect to the Euclidean metric. For $a=1$, we consider the ball $B(x,r)$ with respect to the metric $d(x,y)=\sqrt{\|x\|+\|y\|-\sqrt{2(\|x\|\,\|y\|+\langle x,y\rangle)}}$. The measure $m_{k,1}$ satisfies the scaling property

$$m_{k,1}\left(B\left(tx,\sqrt{t}r\right)\right)=t^{2\langle k\rangle+N-1}m_{k,1}\left(B(x,r)\right),\ t>0. \tag{5.2}$$

From polar coordinate transformation $x=r\omega$, $r>0$, $\omega\in\mathbb{S}^{N-1}$, we have

$$\begin{aligned} m_{k,1}\left(B(x,r)\right) &= \int_{\mathbb{S}^{N-1}}\int_{\substack{(0,+\infty)\\ \rho+\|x\|-2\sqrt{\rho\|x\|}\cos\frac{\theta}{2}\le r^2}} \rho^{2\langle k\rangle+N-2}d\rho h_k(\omega)\,d\omega \\ &\overset{u=\sqrt{\rho}}{=} \int_{\mathbb{S}^{N-1}}\int_{\substack{(0,+\infty)\\ u^2+\|x\|-2u\sqrt{\|x\|}\cos\frac{\theta}{2}\le r^2}} u^{2(2\langle k\rangle+N)-3}du h_k(\omega)\,d\omega \\ &\overset{z=u\omega}{=} \int_{E(x_\omega,r)} \|z\|^{2\langle k\rangle+N-2}\,h_k(z)\,dz, \end{aligned}$$

where

$$\theta=\arccos\frac{\langle x,\omega\rangle}{\|x\|},\ x_\omega=\sqrt{\|x\|}\frac{x+\|x\|\,\omega}{\|x+\|x\|\,\omega\|},$$

and $E(x_\omega,r)$ denotes the Euclidean ball centered at x_ω with radius r. For the one dimensional case, this expression coincides with that of the measure of the ball in the proof of Lemma 2.2 in [6]. So if $2\langle k\rangle+N-2>0$, then for any $x\in\mathbb{R}^N$ and $r>0$, $m_{k,1}(B(x,r))$ is finite and $m_{k,1}(B(tx,r))$ is nondecreasing as t grows. It is then easy to check that $m_{k,1}$ is a doubling measure when $2\langle k\rangle+N-2>0$ combining (5.2). Therefore, for $a=1$ and 2, $(\mathbb{R}^N,d,m_{k,a})$ are both spaces of homogeneous type, and for all $f\in L^1\left(\mathbb{R}^N,\vartheta_{k,a}(x)\,dx\right)\cap L^2\left(\mathbb{R}^N,\vartheta_{k,a}(x)\,dx\right)$

and $\lambda > 0$, there exists the corresponding Calderón–Zygmund decomposition of f satisfying (i)–(v).

Now we are ready to give the Hörmander type condition adapted to (k, a)-generalized setting for $a = 1$ or 2. The classical Hörmander type condition on a homogeneous space in [13, Chapter III, Theorem 2.4] no longer holds in the (k, a)-generalized Fourier analysis, and so we need a modification via the distance of orbits $d_G(x, y)$. The proof is borrowed from that of Theorem 3.1 in [2].

Theorem 5.1. *(See [2] for $a = 2$ and [40] for $a = 1$) For $2\langle k\rangle + N + a - 3 > 0$, $a = 1$ or 2, let K be a measurable function on $\mathbb{R}^N \times \mathbb{R}^N \setminus \left\{(x, g.x); x \in \mathbb{R}^N,\ g \in G\right\}$ and S be a bounded operator on $L^2\left(\mathbb{R}^N, \vartheta_{k,a}(x)\,dx\right)$ associated with the kernel K such that for any compactly supported function $f \in L^2\left(\mathbb{R}^N, \vartheta_{k,a}(x)\,dx\right)$,*

$$S(f)(x) = \int_{\mathbb{R}^N} K(x, y)\, f(y)\, \vartheta_{k,a}(y)\, dy,\ G.x \cap \operatorname{supp} f = \varnothing.$$

If K satisfies

$$\int_{d_G(x,y)>2d(y,y_0)} |K(x,\ y) - K(x,\ y_0)|\, \vartheta_{k,a}(x)\, dx \le C,\quad y, y_0 \in \mathbb{R}^N,$$

then S extends to a bounded operator on $L^p\left(\mathbb{R}^N, \vartheta_{k,a}(x)\,dx\right)$ for $1 < p \le 2$ and a weakly bounded operator on $L^1\left(\mathbb{R}^N, \vartheta_{k,a}(x)\,dx\right)$.

Proof. We will show that S is weakly bounded on $L^1\left(\mathbb{R}^N, \vartheta_{k,a}(x)\,dx\right)$ and conclude by Marcinkiewicz interpolation. The proof is similar to that of Theorem 3.1 in [2] but we repeat it here for reader's convenience.

The proof consists in showing the following inequality on weakly L^1-boundedness for $f = h$ and $f = b$:

$$\rho_\lambda(S(f)) := m_{k,a}\Big(\{x \in \mathbb{R}^N;\ |S(f)(x)| > \frac{\lambda}{2}\}\Big) \le C\, \frac{\|f\|_{L^1\left(\mathbb{R}^N, \vartheta_{k,a}(x)dx\right)}}{\lambda}. \tag{5.3}$$

By using the L^2-boundedness of S, we get

$$\rho_\lambda(S(h)) \le \frac{4}{\lambda^2} \int_{\mathbb{R}^N} |S(h)(x)|^2 dm_{k,a}(x) \le \frac{C}{\lambda^2} \int_{\mathbb{R}^N} |h(x)|^2 dm_{k,a}(x). \tag{5.4}$$

From (i) and (v),

$$\int_{\cup B_j} |h(x)|^2 dm_{k,a}(x) \le C\lambda^2 m_{k,a}(\cup B_j) \le C\lambda\, \|f\|_{L^1\left(\mathbb{R}^N, \vartheta_{k,a}(x)dx\right)}. \tag{5.5}$$

Since on $(\bigcup_j B_j)^c$, $f(x) = h(x)$, then

$$\int_{(\cup_j B_j)^c} |h(x)|^2 dm_{k,a}(x) \leq C\lambda \|f\|_{L^1\left(\mathbb{R}^N,\vartheta_{k,a}(x)dx\right)}. \tag{5.6}$$

From (5.4), (5.5) and (5.6), the inequality (5.3) holds for h.

Next we turn to the inequality (5.3) for the function b. Consider

$$B_j^* = B(y_j, 2r_j\ ); \qquad \text{and} \quad Q_j^* = \bigcup_{g\in G} g.B_j^*.$$

Then

$$\rho_\lambda(S(b)) \leq m_{k,a}\Big(\bigcup_j Q_j^*\Big) + m_{k,a}\left\{x \in \Big(\bigcup_j Q_j^*\Big)^c; |S(b)(x)| > \frac{\lambda}{2}\right\}.$$

Now by (5.1) and *(v)*

$$m_{k,a}\Big(\bigcup_j Q_j^*\Big) \leq |G| \sum_j m_{k,a}(B_j^*) \leq C \sum_j m_{k,a}(B_j) \leq C\ \frac{\|f\|_{L^1\left(\mathbb{R}^N,\vartheta_{k,a}(x)dx\right)}}{\lambda}.$$

Furthermore, if $x \notin Q_j^*$, we have

$$d_G\left(x, y_j\right) > 2d\left(y, y_j\right), \quad y \in B_j.$$

Thus, from *(iii)* ,*(ii)*, *(iv)* and *(v)*

$$\begin{aligned}
&\int_{(\cup Q_j^*)^c} |S(b)(x)| dm_{k,a}(x) \\
\leq&\ \sum_j \int_{(Q_j^*)^c} |S(b_j)(x)| dm_{k,a}(x) \\
=&\ \sum_j \int_{(Q_j^*)^c} \left|\int_{\mathbb{R}^N} K(x,y) b_j(y) dm_{k,a}(y)\right| dm_{k,a}(x) \\
=&\ \sum_j \int_{(Q_j^*)^c} \left|\int_{\mathbb{R}^N} b_j(y)\Big(K(x,y) - K(x,y_j)\Big) dm_{k,a}(y)\right| dm_{k,a}(x) \\
\leq&\ \sum_j \int_{\mathbb{R}^N} |b_j(y)| \int_{(Q_j^*)^c} \left|K(x,y) - K(x,y_j)\right| dm_{k,a}(x) dm_{k,a}(y) \\
\leq&\ \sum_j \int_{\mathbb{R}^N} |b_j(y)| \int_{d_G(x,y_j)>2d(y,y_j)} |K(x,y) - K(x,y_j)| dm_{k,a}(x) dm_{k,a}(y) \\
\leq&\ C \sum_j \|b_j\|_{L^1\left(\mathbb{R}^N,\vartheta_{k,a}(x)dx\right)}
\end{aligned}$$

$$\leq \; C \; \|f\|_{L^1\left(\mathbb{R}^N, \vartheta_{k,a}(x)dx\right)} .$$

Therefore,

$$m_{k,a}\left\{x \in \left(\bigcup_j Q_j^*\right)^c ; |S(b)(x)| > \frac{\lambda}{2}\right\} \leq \frac{2}{\lambda}\int_{(\cup Q_j^*)^c} |S(b)(x)| dm_{k,a}(x) \leq C \; \frac{\|f\|_{L^1\left(\mathbb{R}^N, \vartheta_{k,a}(x)dx\right)}}{\lambda}.$$

This achieves the proof of the inequality (5.3) for b. □

6 Imaginary powers of the (k,a)-generalized harmonic oscillator

We will define and investigate the imaginary powers $(-\triangle_{k,a})^{-i\sigma}, \sigma \in \mathbb{R}$ of the (k,a)-generalized harmonic oscillator $\triangle_{k,a} = \|x\|^{2-a}\triangle_k - \|x\|^a$ and prove the L^p-boundedness $(1 < p < \infty)$ and weak L^1-boundedness of such operators for $a = 2$ and 1 respectively. We will use the Calderón–Zygmund theory developed in the last Chapter to prove such results. For the case of $a = 2$, the result was already proved for $G = \mathbb{Z}_2^N$ in [32]. For general finite reflection groups G in this case, the proof goes similar to that of the L^p-boundedness $(1 < p < \infty)$ and weakly L^1-boundedness of the Riesz transforms related to the Dunkl harmonic oscillator in [1], using the estimate of the derivative of integral kernel. For the case of $a = 1$, however, the method on the estimate of the derivative can no longer be used due to the metric corresponding to the context of $(k,1)$-generalized analysis. For a general metric space, a well-known definition of differentiation by Cheeger [10] is given via integration on continuous rectifiable curves. Unfortunately, as Remark 4.4. ii shows, rectifiable curves between two distinct points do not necessarily exist (or in other words, the induced length metric could be infinite) with respect to the metric corresponding to $(k,1)$-generalized analysis and derivatives on the metric space cannot be defined. We will make use of an estimate of difference quotient analogue in substitute of estimate of derivative. The result and proof are contained for this case in my paper [40]. We denote C, C_1, C_2 to be constants varying from line to line and b, c, b_1, b_2 to be some positive absolute constants in the following.

From the formula

$$\tanh\ \tfrac{z}{2} + \frac{1}{\sinh\ z} = \coth\ z$$

we reformulate the reproducing kernel $\Lambda_{k,a}(x,y;z)$ of $e^{\frac{z}{a}\triangle_{k,a}}$ from (2.13), (4.8) and (4.6) as

$$\begin{aligned}\Lambda_{k,a}(x,y;z) &= \frac{\exp(-\frac{1}{a}(\|x\|^a+\|y\|^a)\tanh\frac{z}{2})}{\sinh(z)^{\frac{2\langle k\rangle+N+a-2}{a}}}\tau_y\left(e^{-\frac{1}{a\sinh z}\|\cdot\|^a}\right)((-1)^{\frac{2}{a}}x) \qquad (6.1)\\ &= \frac{\exp(-\frac{1}{a}(\|x\|^a+\|y\|^a)\tanh\frac{z}{2})}{\sinh(z)^{\frac{2\langle k\rangle+N+a-2}{a}}}\end{aligned}$$

$$\times \begin{cases} \dfrac{\Gamma\left(\frac{N-1}{2}+\langle k\rangle\right)}{\sqrt{\pi}\Gamma\left(\frac{N-2}{2}+\langle k\rangle\right)}\times \\ V_k\Bigg(\int_{-1}^{1} e^{-\frac{1}{\sinh z}\left(\|x\|+\|y\|-\sqrt{2(\|x\|\|y\|+\langle\cdot,y\rangle)}u\right)}\left(1-u^2\right)^{\frac{N}{2}+\langle k\rangle-2}du\Bigg)(x) & (a=1), \\ V_k\left(e^{-\frac{1}{2\sinh z}\left(\|x\|^2+\|y\|^2-2\langle\cdot,y\rangle\right)}\right)(x) & (a=2). \end{cases} \tag{6.2}$$

Let $z=at$, $t>0$. For $0<t\leq 1$, $\sinh at$ behaves like at. So

$$\left|\Lambda_{k,a}\left(x,y;at\right)\right| \leq C\frac{1}{t^{\frac{2\langle k\rangle+N+a-2}{a}}}\tau_y\left(e^{-\frac{b}{t}\|\cdot\|^a}\right)((-1)^{\frac{2}{a}}x). \tag{6.3}$$

For $t>1$, $\sinh at$ behaves like e^{at}. So

$$\left|\Lambda_{k,a}\left(x,y;at\right)\right| \leq Ce^{-(2\langle k\rangle+N+a-2)t}\tau_y\left(e^{-b\|\cdot\|^a}\right)((-1)^{\frac{2}{a}}x). \tag{6.4}$$

From (2.6) we can define the imaginary powers $\left(-\triangle_{k,a}\right)^{-i\sigma}, \sigma\in\mathbb{R}$ for $f\in L^2(\mathbb{R}^N,$ $\vartheta_{k,a}\left(x\right)dx)$ of the (k,a)-generalized harmonic oscillator $-\triangle_{k,a}$ naturally as

$$\left(-\triangle_{k,a}\right)^{-i\sigma}\left(f\right)\left(x\right)=\sum_{l,m,j}\left(a\left(2l+\lambda_{k,a,m}+1\right)\right)^{-i\sigma}\left\langle f,\Phi_{l,m,j}^{(a)}\right\rangle_{k,a}\Phi_{l,m,j}^{(a)}\left(x\right). \tag{6.5}$$

It is obviously a bounded operator on $L^2\left(\mathbb{R}^N,\vartheta_{k,a}\left(x\right)dx\right)$ from its spectrum.

In what follow we put

$$K\left(x,y\right)=\int_0^\infty \Lambda_{k,a}\left(x,y;at\right)t^{i\sigma-1}dt. \tag{6.6}$$

We will show that the integral (6.6) converges absolutely for $a=1$ or 2. Based on the formula

$$\lambda^{-i\sigma}=\frac{1}{\Gamma\left(i\sigma\right)}\int_0^\infty e^{-t\lambda}t^{i\sigma-1}dt,\quad \lambda>0$$

and (6.5), (2.9), (2.10), we can write $\left(-\triangle_{k,a}\right)^{-i\sigma}$ in the following way (such definition goes back to [32] and [38])

$$\begin{aligned}\left(-\triangle_{k,a}\right)^{-i\sigma}\left(f\right)\left(x\right)&=\frac{1}{\Gamma\left(i\sigma\right)}\int_0^\infty e^{t\triangle_{k,a}}\left(f\right)\left(x\right)t^{i\sigma-1}dt\\&=\frac{c_{k,a}}{\Gamma\left(i\sigma\right)}\int_0^\infty t^{i\sigma-1}dt\int_{\mathbb{R}^N}f\left(y\right)\Lambda_{k,a}\left(x,y;at\right)\vartheta_{k,a}\left(y\right)dy.\end{aligned} \tag{6.7}$$

We will observe that this integral converges absolutely for all compactly supported functions $f\in L^2(\mathbb{R}^N,\vartheta_{k,a}\left(x\right)\,dx)$ with $\operatorname{supp} f\cap G.x=\varnothing$. And for compactly supported functions $f\in L^2\left(\mathbb{R}^N,\vartheta_{k,a}\left(x\right)dx\right)$, $G.x\cap\operatorname{supp} f=\varnothing$, $\left(-\triangle_{k,a}\right)^{-i\sigma}$ satisfies

$$\left(-\triangle_{k,a}\right)^{-i\sigma}\left(f\right)\left(x\right)=\frac{c_{k,a}}{\Gamma\left(i\sigma\right)}\int_{\mathbb{R}^N}K\left(x,y\right)f\left(y\right)\vartheta_{k,a}\left(y\right)dy$$

by changing the order of integration. We will show that the kernel $K(x,y)$ of $(-\triangle_{k,a})^{-i\sigma}$ for $a=1$ or 2 satisfies the condition in Theorem 5.1 to prove the following main theorem.

Theorem 6.1. *([40]) For $2\langle k\rangle+N+a-3>0$, $a=1$ or 2, the imaginary powers $(-\triangle_{k,a})^{-i\sigma}, \sigma\in\mathbb{R}$ of the (k,a)-generalized harmonic oscillator $-\triangle_{k,a}$ are bounded operators on $L^p\left(\mathbb{R}^N,\vartheta_{k,a}(x)\,dx\right)$, $1<p<\infty$, and weakly bounded on $L^1\left(\mathbb{R}^N,\vartheta_{k,a}(x)\,dx\right)$.*

In the following two sections we will prove the above theorem for $a=2$ and 1 respectively. We will show the convergence of the integral (6.7) first.

6.1 The case of $a=2$

Lemma 6.2. *For all $x,y\in\mathbb{R}^N$, $a=2$, $y\notin G.x$, the integral* (6.6) *converges absolutely and*

$$|K(x,y)|\le C\frac{1}{d_G(x,y)^{2\langle k\rangle+N}}.$$

Proof. From (6.3), (4.2), (4.3), we have

$$\begin{aligned}\int_0^1\left|\Lambda_{k,2}(x,y;2t)\,t^{i\sigma-1}\right|dt&\le C\int_0^1\frac{1}{t^{\langle k\rangle+\frac{N}{2}}}\tau_y\left(e^{-\frac{b}{t}\|\cdot\|^2}\right)(-x)\frac{1}{t}dt\\&\le C\int_{\mathbb{R}^N}\int_0^1\frac{1}{t^{\langle k\rangle+\frac{N}{2}+1}}e^{-\frac{b}{t}\left(\|x\|^2+\|y\|^2-2\langle\eta,y\rangle\right)}dtd\mu_x(\eta)\\&\le C\int_{\mathbb{R}^N}\frac{1}{\left(\|x\|^2+\|y\|^2-2\langle\eta,y\rangle\right)^{\langle k\rangle+\frac{N}{2}}}d\mu_x(\eta)\int_0^\infty\frac{1}{s^{\langle k\rangle+\frac{N}{2}+1}}e^{-\frac{b}{s}}ds\\&\le C\frac{1}{d_G(x,y)^{2\langle k\rangle+N}}\int_{\mathbb{R}^N}d\mu_x(\eta)\le C\frac{1}{d_G(x,y)^{2\langle k\rangle+N}}.\end{aligned}$$

And from (6.4), (4.2), (4.3),

$$\begin{aligned}\int_1^\infty\left|\Lambda_{k,2}(x,y;2t)\,t^{i\sigma-1}\right|dt&\le C\int_1^\infty e^{-(2\langle k\rangle+N)t}\tau_y\left(e^{-b\|\cdot\|^2}\right)(-x)\frac{1}{t}dt\\&\le C\tau_y\left(e^{-b\|\cdot\|^2}\right)(-x)\le C\int_{\mathbb{R}^N}d\mu_x(\eta)\,e^{-b\cdot d_G(x,y)^2}\\&\le Ce^{-b\cdot d_G(x,y)^2}\le C\frac{1}{d_G(x,y)^{2\langle k\rangle+N}}.\end{aligned}$$

□

Thus the integral (6.7) for $a=2$ converges absolutely for all compactly supported functions $f\in L^2\left(\mathbb{R}^N,h_k(x)\,dx\right)$ with $G.x\cap\operatorname{supp}f=\varnothing$, because

$$\int_{\mathbb{R}^N}\int_0^\infty\left|t^{i\sigma-1}f(y)\,\Lambda_{k,2}(x,y;2t)\right|h_k(y)\,dtdy\le C\int_{\mathbb{R}^N}\frac{1}{d_G(x,y)^{2\langle k\rangle+N}}|f(y)|\,h_k(y)\,dy.$$

We need also an estimate of the partial derivative of the integral kernel $\Lambda_{k,2}(x,y;2t)$ before the proof.

Lemma 6.3. *For $0 < t < 1$, we have*

$$\left|\frac{\partial \Lambda_{k,2}}{\partial y_i}(x,y;2t)\right| \leq \frac{C}{t^{\langle k\rangle+\frac{N+1}{2}}}\tau_y\left(e^{-\frac{c}{t}\|\cdot\|^2}\right)(-x).$$

Proof. From (6.1), we write

$$\begin{aligned}\frac{\partial \Lambda_{k,2}}{\partial y_i}(x,y;2t) &= \frac{1}{\sinh(2t)^{\langle k\rangle+\frac{N}{2}}}\Big(-\tau_{-y}\left(e^{-\frac{1}{2\sinh 2t}\|\cdot\|^2}\right)(x)e^{-\frac{1}{2}(\|x\|^2+\|y\|^2)\tanh t}y_i\tanh t\\ &\quad - e^{-\frac{1}{2}(\|x\|^2+\|y\|^2)\tanh t}V_k\left(e^{-\frac{1}{2\sinh 2t}(\|x\|^2+\|y\|^2-2\langle\cdot,y\rangle)}\frac{1}{2\sinh 2t}\left(y_i-(\cdot)_i\right)\right)(x)\Big)\\ &= \frac{1}{\sinh(2t)^{\langle k\rangle+\frac{N}{2}}}\left(I_1+I_2\right).\end{aligned}$$

Notice that $\sinh 2t$ behaves like $2t$ for $0 < t < 1$. For the first part I_1, along with the fact that $ue^{-\frac{1}{2}u^2}$ is a bounded function, we have

$$|I_1| \leq C_1\sqrt{t}\tau_y\left(e^{-\frac{b_1}{t}\|\cdot\|^2}\right)(-x).$$

For the second part I_2, we have

$$\begin{aligned}|I_2| &\leq \int_{co(G.x)} e^{-\frac{1}{2\sinh 2t}(\|x\|^2+\|y\|^2-2\langle\eta,y\rangle)}\frac{1}{2\sinh 2t}|y_i-\eta_i|\,d\mu_x(\eta)\\ &\leq \int_{co(G.x)} e^{-\frac{1}{4\sinh 2t}(\|x\|^2+\|y\|^2-2\langle\eta,y\rangle)}e^{-\frac{1}{4\sinh 2t}(\|\eta\|^2+\|y\|^2-2\langle\eta,y\rangle)}\frac{1}{2\sinh 2t}\|y-\eta\|\,d\mu_x(\eta)\\ &\leq C_2\frac{1}{\sqrt{\sinh 2t}}\int_{co(G.x)} e^{-\frac{1}{4\sinh 2t}(\|x\|^2+\|y\|^2-2\langle\eta,y\rangle)}d\mu_x(\eta)\\ &\leq C_2\frac{1}{\sqrt{t}}\tau_{-y}\left(e^{-\frac{b_2}{t}\|\cdot\|^2}\right)(x).\end{aligned}$$

Thus

$$\begin{aligned}\left|\frac{\partial \Lambda_{k,2}}{\partial y_i}(x,y;2t)\right| &\leq \frac{1}{t^{\langle k\rangle+\frac{N}{2}}}\left(C_1\sqrt{t}+C_2\frac{1}{\sqrt{t}}\right)\tau_y\left(e^{-\frac{c}{t}\|\cdot\|^2}\right)(-x)\\ &\leq \frac{C}{t^{\langle k\rangle+\frac{N+1}{2}}}\tau_y\left(e^{-\frac{c}{t}\|\cdot\|^2}\right)(-x).\end{aligned}$$ □

We can then start the proof of Theorem 6.1 for $a = 2$.

Proof of Theorem 6.1 (for $a = 2$). We only need to show that the operator $(-\triangle_{k,2})^{-i\sigma}$ is L^p-bounded for $1 < p \leq 2$ and weakly L^1-bounded since it is symmetric on $L^2\left(\mathbb{R}^N, \vartheta_{k,2}(x)\,dx\right)$ and its L^p-boundedness for $2 < p < \infty$ can be derived from the duality argument. From (6.6), we write

$$K(x,y) = \int_0^1 \Lambda_{k,2}(x,y;2t)\,t^{i\sigma-1}dt + \int_1^\infty \Lambda_{k,2}(x,y;2t)\,t^{i\sigma-1}dt$$

$$= K^{(1)}(x,y) + K^{(2)}(x,y),$$

where $x, y \in \mathbb{R}^N$, $y \notin G.x$. We claim that $K(x,y)$ satisfies the condition in Theorem 5.1.

For the second part $K^{(2)}(x,y)$, by (6.3), (4.1) and the Property 2 of Dunkl translations,

$$\begin{aligned}\int_{\mathbb{R}^N} \left|K^{(2)}(x,y)\right| \vartheta_{k,2}(x)\,dx &\leq C \int_{\mathbb{R}^N}\int_1^\infty e^{-(2\langle k\rangle+N)t}\tau_y\left(e^{-b\|\cdot\|}\right)(x)\frac{1}{t}h_k(x)\,dtdx \\ &= C\int_1^\infty \int_{\mathbb{R}^N} e^{-(2\langle k\rangle+N)t}e^{-b\|x\|}\frac{1}{t}h_k(x)\,dx \\ &\leq C\int_1^\infty e^{-(2\langle k\rangle+N)t}\frac{1}{t}dt \leq C.\end{aligned}$$

Then we have

$$\int_{d_G(x,y)>2\|y-y_0\|}\left|K^{(2)}(x,y) - K^{(2)}(x,\ y_0)\right| h_k(x)\,dx \leq 2\int_{\mathbb{R}^N}\left|K^{(2)}(x,y)\right| h_k(x)\,dx \leq C.$$

For the first part $K^{(1)}(x,y)$, from Lemma 6.3,

$$\begin{aligned}\left|K^{(1)}(x,y) - K^{(1)}(x,y_0)\right| &\leq \int_0^1 \left|\Lambda_{k,2}(x,y;2t) - \Lambda_{k,2}(x,y_0;2t)\right|\frac{1}{t}dt \\ &\leq \|y-y_0\|\int_0^1 \frac{1}{t}dt\int_0^1\sum_{i=1}^N\left|\frac{\partial\Lambda_{k,2}}{\partial y_i}(x,y_\theta,;2t)\right|d\theta \\ &\leq C\|y-y_0\|\int_0^1\frac{1}{t^{\langle k\rangle+\frac{N+3}{2}}}dt\int_0^1\tau_x\left(e^{-\frac{c}{t}\|\cdot\|^2}\right)(-y_\theta)d\theta,\end{aligned}$$

where $y_\theta = y_0 + \theta(y-y_0)$. When $d_G(x,y) > 2\|y-y_0\|$, we have

$$d_G(x,y_\theta) \geq d_G(x,y) - \|y-y_\theta\| > \|y-y_0\|.$$

Then from (4.2) and (4.3), we have

$$\tau_x\left(e^{-\frac{c}{t}\|\cdot\|^2}\right)(-y_\theta) \leq \tau_x\left(e^{-\frac{c}{4t}(\|\cdot\|+\|y-y_0\|)^2}\right)(-y_\theta).$$

Therefore, from the Property 2 of Dunkl translations and (4.1),

$$\begin{aligned}&\int_{d_G(x,y)>2\|y-y_0\|}\left|K^{(1)}(x,y) - K^{(1)}(x,y_0)\right| h_k(x)\,dx \\ &\leq C\|y-y_0\|\int_0^1\frac{1}{t^{\langle k\rangle+\frac{N+3}{2}}}dt\int_0^1\left(\int_{\mathbb{R}^N}\tau_{y_\theta}\left(e^{-\frac{c}{4t}(\|\cdot\|+\|y-y_0\|)^2}\right)(-x)h_k(x)\,dx\right)d\theta \\ &= C\|y-y_0\|\int_0^1\frac{1}{t^{\langle k\rangle+\frac{N+3}{2}}}dt\int_{\mathbb{R}^N}e^{-\frac{c}{4t}(\|x\|+\|y-y_0\|)^2}h_k(x)\,dx \\ &= C\|y-y_0\|\int_0^\infty r^{2\langle k\rangle+N-1}dr\int_0^1\frac{1}{t^{\langle k\rangle+\frac{N+3}{2}}}e^{-\frac{c}{4t}(r+\|y-y_0\|)^2}dt\end{aligned}$$

$$\leq C\left\|y-y_0\right\| \int_0^{\infty} \frac{r^{2\langle k\rangle+N-1}}{\left(r+\left\|y-y_0\right\|\right)^{2\langle k\rangle+N+1}} dr \int_0^{\infty} \frac{1}{u^{\langle k\rangle+\frac{N+3}{2}}} e^{-\frac{c}{4u}} du$$
$$\leq C\left\|y-y_0\right\| \int_0^{\infty} \frac{1}{\left(r+\left\|y-y_0\right\|\right)^2} dr=C.$$

The proof of Theorem 6.1 is complete for $a = 2$. □

6.2 The case of $a = 1$

Lemma 6.4. *For all $x, y \in \mathbb{R}^N$, $a = 1$, $y \notin G.x$, $2\langle k\rangle + N - 2 > 0$, the integral* (6.6) *converges absolutely and*

$$|K(x, y)| \leq C \frac{1}{d_G(x, y)^{2(2\langle k\rangle+N-1)}}.$$

Proof. From (6.3), (4.6), (4.9), we have

$$\begin{aligned}
\int_0^1\left|\Lambda_{k,1}(x, y; t) t^{i\sigma-1}\right| dt & \leq C \int_0^1 \frac{1}{t^{2\langle k\rangle+N-1}} \tau_y\left(e^{-\frac{b}{t}\|\cdot\|}\right)(x) \frac{1}{t} dt \\
& \leq C \int_{\mathbb{R}^N} \int_{-1}^1 \int_0^1 \frac{1}{t^{2\langle k\rangle+N}} e^{-\frac{b}{t}\left(\|x\|+\|y\|-\sqrt{2(\|x\|\|y\|+\langle\eta, y\rangle) u}\right)} dt \\
& \quad \left(1-u^2\right)^{\frac{N}{2}+\langle k\rangle-2} du d\mu_x(\eta) \\
& \leq C \int_{\mathbb{R}^N} d\mu_x(\eta) \int_{-1}^1 \frac{1}{\left(\|x\|+\|y\|-\sqrt{2(\|x\|\|y\|+\langle\eta, y\rangle) u}\right)^{2\langle k\rangle+N-1}} \\
& \quad \cdot\left(1-u^2\right)^{\frac{N}{2}+\langle k\rangle-2} du \int_0^{\infty} \frac{1}{s^{2\langle k\rangle+N}} e^{-\frac{b}{s}} ds \\
& \leq C \frac{1}{d_G(x, y)^{2(2\langle k\rangle+N-1)}} \int_{\mathbb{R}^N} d\mu_x(\eta) \int_{-1}^1\left(1-u^2\right)^{\frac{N}{2}+\langle k\rangle-2} du \\
& \leq C \frac{1}{d_G(x, y)^{2(2\langle k\rangle+N-1)}}.
\end{aligned}$$

And from (6.4), (4.6), (4.9),

$$\begin{aligned}
\int_1^{\infty}\left|\Lambda_{k,1}(x, y; t) t^{i\sigma-1}\right| dt & \leq C \int_1^{\infty} e^{-(2\langle k\rangle+N-1) t} \tau_y\left(e^{-b\|\cdot\|}\right)(x) \frac{1}{t} dt \\
& \leq C \tau_y\left(e^{-b\|\cdot\|}\right)(x) \\
& \leq C \int_{\mathbb{R}^N} d\mu_x(\eta) \int_{-1}^1\left(1-u^2\right)^{\frac{N}{2}+\langle k\rangle-2} du \cdot e^{-b \cdot d_G(x, y)^2} \\
& \leq C e^{-b \cdot d_G(x, y)^2} \leq C \frac{1}{d_G(x, y)^{2(2\langle k\rangle+N-1)}}.
\end{aligned}$$
□

Thus the integral (6.7) for $a = 1$ converges absolutely for all compactly supported functions $f \in L^2\left(\mathbb{R}^N, \vartheta_{k,1}(x)\,dx\right)$ with $G.x \cap \operatorname{supp} f = \varnothing$, because

$$\int_{\mathbb{R}^N}\int_0^{\infty}\left|t^{i\sigma-1} f(y)\,\Lambda_{k,1}(x,y;t)\right|\vartheta_{k,1}(y)\,dtdy \le C\int_{\mathbb{R}^N}\frac{1}{d_G(x,y)^{2(2\langle k\rangle+N-1)}}\left|f(y)\right|\vartheta_{k,1}(y)\,dy.$$

We can then start the proof of Theorem 6.1 for $a = 1$. It begins with the following two lemmas. The first one is an enhancement of the triangle inequality of the metric $d(x, y)$.

Lemma 6.5. *For $u \in [-1, 1]$, $\eta \in co(G.x)$, and $x, y \in \mathbb{R}^N$,*

$$\left|\sqrt{\|x\|+\|y\|-\sqrt{2\left(\|x\|\,\|y\|+\langle\eta,y\rangle\right)u}}-\sqrt{\|x\|+\|z\|-\sqrt{2\left(\|x\|\,\|z\|+\langle\eta,z\rangle\right)u}}\right| \le d(y,z).$$

Proof. If $\eta \in co(G.x)$, then there exists a rotation transformation T such that $\eta = kT(x)$, $0 \le k \le 1$. So we assume $\eta = kx$ in the proof since $\|T(x)\| = \|x\|$. Then

$$\sqrt{2\left(\|x\|\,\|y\|+\langle\eta,y\rangle\right)u} = 2\sqrt{\|x\|\,\|y\|}\sqrt{\frac{1}{2}\left(1+k\cos\alpha\right)u};$$

$$\sqrt{2\left(\|x\|\,\|z\|+\langle\eta,z\rangle\right)u} = 2\sqrt{\|x\|\,\|z\|}\sqrt{\frac{1}{2}\left(1+k\cos\beta\right)u}.$$

Denote by

$$\alpha_k = 2\arccos\sqrt{\frac{1}{2}\left(1+k\cos\alpha\right)},\quad \beta_k = 2\arccos\sqrt{\frac{1}{2}\left(1+k\cos\beta\right)}.$$

We assert that

$$|\alpha_k - \beta_k| \le \gamma. \tag{6.8}$$

Here α, β and γ are given as in the proof of Proposition 4.3.

Assume $\|x\| = \|y\| = \|z\| = 1$. Then (6.8) is equivalent to

$$1-\langle y,z\rangle^2-k^2\langle x,y\rangle^2-k^2\langle x,z\rangle^2+2k^2\langle x,y\rangle\langle y,z\rangle\langle x,z\rangle \ge 0.$$

It suffices to show that

$$k^2\left(1-\langle y,z\rangle^2\right)-k^2\langle x,y\rangle^2-k^2\langle x,z\rangle^2+2k^2\langle x,y\rangle\langle y,z\rangle\langle x,z\rangle \ge 0.$$

And it is equivalent to

$$\det\begin{bmatrix}1 & \langle x,y\rangle & \langle x,z\rangle\\ \langle y,x\rangle & 1 & \langle y,z\rangle\\ \langle z,x\rangle & \langle z,y\rangle & 1\end{bmatrix} \ge 0.$$

It is the determinant of the Gram matrix of the three vectors x, y, and z. Thus (6.8) is proved.

From assertion (6.8), similar to the proof in Proposition 4.3, it suffices to show that

$$\left|\sqrt{\|x\| + \|y\| - 2\sqrt{\|x\|\,\|y\|}u\cos\frac{\alpha_k}{2}} - \sqrt{\|x\| + \|z\| - 2\sqrt{\|x\|\,\|z\|}u\cos\frac{\beta_k}{2}}\right| \leq \sqrt{\|y\| + \|z\| - 2\sqrt{\|y\|\,\|z\|}\cos\frac{\alpha_k - \beta_k}{2}}.$$

And it suffices to show

$$\begin{aligned}\|x\|\,\|y\|\left(1 - u^2\cos^2\frac{\alpha_k}{2}\right) + \|x\|\,\|z\|\left(1 - u^2\cos^2\frac{\beta_k}{2}\right) + \|z\|\,\|y\|\sin^2\frac{\alpha_k - \beta_k}{2} \\ + 2\,\|z\|\,\sqrt{\|x\|\,\|y\|}u\cos\frac{\beta_k}{2}\cos\frac{\alpha_k - \beta_k}{2} \\ +2\,\|x\|\,\sqrt{\|z\|\,\|y\|}u^2\cos\frac{\beta_k}{2}\cos\frac{\alpha_k}{2} + 2\,\|y\|\,\sqrt{\|z\|\,\|x\|}u\cos\frac{\alpha_k - \beta_k}{2}\cos\frac{\alpha_k}{2} \\ \geq 2\,\|x\|\,\sqrt{\|z\|\,\|y\|}\cos\frac{\alpha_k - \beta_k}{2} + 2\,\|y\|\,\sqrt{\|z\|\,\|x\|}u\cos\frac{\beta_k}{2} + 2\,\|z\|\,\sqrt{\|x\|\,\|y\|}u\cos\frac{\alpha_k}{2}.\end{aligned}$$

The above is equivalent to

$$\left(\sqrt{\|x\|\,\|y\|}u\sin\frac{\alpha_k}{2} - \sqrt{\|x\|\,\|z\|}u\sin\frac{\beta_k}{2} - \sqrt{\|z\|\,\|y\|}\sin\frac{\alpha_k - \beta_k}{2}\right)^2 + \|x\|\left(\|y\| + \|z\|\right)\left(1 - u^2\right) \geq 2\,\|x\|\,\sqrt{\|z\|\,\|y\|}\left(1 - u^2\right)\cos\frac{\alpha_k - \beta_k}{2}.$$

The Lemma is therefore proved. □

The next lemma is an estimate of the difference quotient analogue. We can no longer make use of estimates of partial derivatives because we cannot define differentiation on the metric space corresponding to $(k,1)$-generalized analysis for the failure of the existence of continuous rectifiable curves between two distinct points (see Remark 4.4. ii).

Lemma 6.6. *For $0 < t < 1$, $y \neq y_0$,*

$$\left|\frac{\Lambda_{k,1}(x, y; t) - \Lambda_{k,1}(x, y_0; t)}{d(y, y_0)}\right| \leq \frac{C}{t^{2\langle k\rangle + N - \frac{1}{2}}}\left(\tau_{y_0}\left(e^{-\frac{c}{t}\|\cdot\|}\right)(x) + \tau_y\left(e^{-\frac{c}{t}\|\cdot\|}\right)(x)\right).$$

Proof. From (6.1), we write

$$\frac{\Lambda_{k,1}(x, y; t) - \Lambda_{k,1}(x, y_0; t)}{d(y, y_0)}$$

$$= \frac{1}{(\sinh\ t)^{2\langle k\rangle+N-1}} \left(\frac{e^{-\tanh\frac{t}{2}(\|x\|+\|y\|)}\tau_x\left(e^{-\frac{1}{\sinh\ t}\|\cdot\|}\right)(y) - e^{-\tanh\frac{t}{2}(\|x\|+\|y\|)}\tau_x\left(e^{-\frac{1}{\sinh\ t}\|\cdot\|}\right)(y_0)}{d(y,y_0)} \right.$$
$$\left. + \frac{e^{-\tanh\frac{t}{2}(\|x\|+\|y\|)}\tau_x\left(e^{-\frac{1}{\sinh\ t}\|\cdot\|}\right)(y_0) - e^{-\tanh\frac{t}{2}(\|x\|+\|y_0\|)}\tau_x\left(e^{-\frac{1}{\sinh\ t}\|\cdot\|}\right)(y_0)}{d(y,y_0)} \right)$$
$$= \frac{1}{(\sinh\ t)^{2\langle k\rangle+N-1}}(I_1+I_2).$$

Notice that $\sinh\ t$ behaves like t for $0 < t \leq 1$. For the second part I_2, if $\|y\| = \|y_0\|$, then $I_2 = 0$. If $\|y\| \neq \|y_0\|$, then from the inequality

$$\left|\frac{e^{-\tanh\frac{t}{2}\cdot x_1^2} - e^{-\tanh\frac{t}{2}\cdot x_2^2}}{x_1 - x_2}\right| \leq \max_x \left|2\tanh\frac{t}{2}\cdot xe^{-\tanh\frac{t}{2}\cdot x^2}\right| \leq C_2\sqrt{t},$$

we have

$$|I_2| = \left|\tau_x\left(e^{-\frac{1}{\sinh\ t}\|\cdot\|}\right)(y_0)\frac{e^{-\tanh\frac{t}{2}(\|x\|+\|y\|)} - e^{-\tanh\frac{t}{2}(\|x\|+\|y_0\|)}}{d(y,y_0)}\right|$$
$$\leq \tau_x\left(e^{-\frac{1}{\sinh\ t}\|\cdot\|}\right)(y_0)\cdot\left|\frac{e^{-\tanh\frac{t}{2}\|y\|} - e^{-\tanh\frac{t}{2}\|y_0\|}}{\sqrt{\|y\|} - \sqrt{\|y_0\|}}\right|$$
$$\leq C_2\sqrt{t}\tau_x\left(e^{-\frac{b_2}{t}\|\cdot\|}\right)(y_0).$$

For the first part I_1, from the inequality

$$\left|\frac{e^{-\frac{1}{\sinh\ t}\cdot x_1^2} - e^{-\frac{1}{\sinh\ t}\cdot x_2^2}}{x_1 - x_2}\right| \leq \max_{x_2\leq x\leq x_1}\left|\frac{2}{\sinh\ t}\cdot xe^{-\frac{1}{\sinh\ t}\cdot x^2}\right|$$
$$\leq \frac{2}{\sqrt{\sinh\ t}}e^{-\frac{1}{2\sinh\ t}\cdot x_2^2}\max_{x_2\leq x\leq x_1}\left|\frac{1}{\sqrt{\sinh\ t}}\cdot xe^{-\frac{1}{2\sinh\ t}\cdot x^2}\right|$$
$$\leq C_1\frac{1}{\sqrt{t}}e^{-\frac{b_1}{t}\cdot x_2^2}, \quad x_1 > x_2,$$

along with Lemma 6.5 and (4.6),

$$|I_1| \leq C_1 e^{-\tanh\frac{t}{2}(\|x\|+\|y\|)}$$
$$\cdot V_k\left(\int_{-1}^{1}\left|\frac{e^{-\frac{1}{\sinh\ t}\left(\|x\|+\|y\|-\sqrt{2(\|x\|\|y\|+\langle\cdot,y\rangle)u}\right)} - e^{-\frac{1}{\sinh\ t}\left(\|x\|+\|y_0\|-\sqrt{2(\|x\|\|y_0\|+\langle\cdot,y_0\rangle)u}\right)}}{d(y,y_0)}\right|\right.$$
$$\left.\cdot\left(1-u^2\right)^{\frac{N}{2}+\langle k\rangle-2}du\right)(x)$$
$$\leq \frac{C_1}{\sqrt{t}}e^{-\tanh\frac{t}{2}(\|x\|+\|y\|)}\int_{\mathbb{R}^N}\left(\int_{\left\{u\in[-1,1]:\|y\|-\sqrt{2(\|x\|\|y\|+\langle\eta,y\rangle)u}>\|y_0\|-\sqrt{2(\|x\|\|y_0\|+\langle\eta,y_0\rangle)u}\right\}}\right.$$

$$e^{-\frac{1}{2\sinh t}\left(\|x\|+\|y_0\|-\sqrt{2(\|x\|\|y_0\|+\langle\eta,y_0\rangle)}u\right)}\left(1-u^2\right)^{\frac{N}{2}+\langle k\rangle-2}du$$
$$+\int_{\left\{u\in[-1,1]:\|y\|-\sqrt{2(\|x\|\|y\|+\langle\eta,y\rangle)}u<\|y_0\|-\sqrt{2(\|x\|\|y_0\|+\langle\eta,y_0\rangle)}u\right\}}e^{-\frac{1}{2\sinh t}\left(\|x\|+\|y\|-\sqrt{2(\|x\|\|y\|+\langle\eta,y\rangle)}u\right)}$$
$$\left.\cdot\left(1-u^2\right)^{\frac{N}{2}+\langle k\rangle-2}du\right)d\mu_x(\eta)$$
$$\leq\frac{C_1}{\sqrt{t}}\left(\tau_{y_0}\left(e^{-\frac{b_1}{t}\|\cdot\|}\right)(x)+\tau_y\left(e^{-\frac{b_1}{t}\|\cdot\|}\right)(x)\right).$$

Thus

$$\left|\frac{\Lambda_{k,1}(x,y;t)-\Lambda_{k,1}(x,y_0;t)}{d(y,y_0)}\right|\leq\frac{1}{t^{2\langle k\rangle+N-1}}\left(C_2\sqrt{t}+\frac{C_1}{\sqrt{t}}\right)\left(\tau_{y_0}\left(e^{-\frac{c}{t}\|\cdot\|}\right)(x)+\tau_y\left(e^{-\frac{c}{t}\|\cdot\|}\right)(x)\right)$$
$$\leq\frac{C}{t^{2\langle k\rangle+N-\frac{1}{2}}}\left(\tau_{y_0}\left(e^{-\frac{c}{t}\|\cdot\|}\right)(x)+\tau_y\left(e^{-\frac{c}{t}\|\cdot\|}\right)(x)\right).\qquad\square$$

Proof of Theorem 6.1 (for $a = 1$). We only need to show that the operator $(-\triangle_{k,1})^{-i\sigma}$ is L^p-bounded for $1 < p \leq 2$ and weakly L^1-bounded since it is symmetric on $L^2\left(\mathbb{R}^N,\vartheta_{k,1}(x)\,dx\right)$ and its L^p-boundedness for $2 < p < \infty$ can be derived from the duality argument. From (6.6), we write

$$K(x,y)=\int_0^1\Lambda_{k,1}(x,y;t)\,t^{i\sigma-1}dt+\int_1^\infty\Lambda_{k,1}(x,y;t)\,t^{i\sigma-1}dt$$
$$=K^{(1)}(x,y)+K^{(2)}(x,y),$$

where $x,y\in\mathbb{R}^N$, $y\notin G.x$. We claim that $K(x,y)$ satisfies the condition in Theorem 5.1.

For the second part $K^{(2)}(x,y)$, by (6.3), (4.4) and (4.7),

$$\int_{\mathbb{R}^N}\left|K^{(2)}(x,y)\right|\vartheta_{k,1}(x)\,dx\leq C\int_{\mathbb{R}^N}\int_1^\infty e^{-(2\langle k\rangle+N-1)t}\tau_y\left(e^{-b\|\cdot\|}\right)(x)\frac{1}{t}\vartheta_{k,1}(x)\,dtdx$$
$$=C\int_1^\infty\int_{\mathbb{R}^N}e^{-(2\langle k\rangle+N-1)t}e^{-b\|x\|}\frac{1}{t}\vartheta_{k,1}(x)\,dx$$
$$\leq C\int_1^\infty e^{-(2\langle k\rangle+N-1)t}\frac{1}{t}dt\leq C.$$

Then we have

$$\int_{d_G(x,y)>2d(y,y_0)}\left|K^{(2)}(x,y)-K^{(2)}(x,\ y_0)\right|\vartheta_{k,1}(x)\,dx\leq 2\int_{\mathbb{R}^N}\left|K^{(2)}(x,y)\right|\vartheta_{k,1}(x)\,dx\leq C.$$

For the first part $K^{(1)}(x,y)$, from Lemma 6.6,

$$\left|K^{(1)}(x,y)-K^{(1)}(x,y_0)\right|\leq\int_0^1\left|\Lambda_{k,1}(x,y;t)-\Lambda_{k,1}(x,y_0;t)\right|\frac{1}{t}dt$$

$$\leq Cd\,(y,y_0)\int_0^1 \frac{1}{t^{2\langle k\rangle+N+\frac{1}{2}}}\left(\tau_{y_0}\left(e^{-\frac{c}{t}\|\cdot\|}\right)(x)+\tau_y\left(e^{-\frac{c}{t}\|\cdot\|}\right)(x)\right)dt.$$

When $d_G(x,y)>2d(y,y_0)$, we have

$$d_G(x,y_0)\geq d_G(x,y)-d(y_0,y)>d(y,y_0),\quad d_G(x,y)>d(y,y_0).$$

Then from (4.9), for any $u\in[-1,1]$ and $\eta\in co(G.x)$, we have

$$\sqrt{\|x\|+\|y_0\|-\sqrt{2\left(\|x\|\,\|y_0\|+\langle\eta,y_0\rangle\right)u}}\geq d_G(x,y_0)>d(y,y_0),$$

$$\sqrt{\|x\|+\|y\|-\sqrt{2\left(\|x\|\,\|y\|+\langle\eta,y\rangle\right)u}}\geq d_G(x,y)>d(y,y_0).$$

So

$$\tau_x\left(e^{-\frac{c}{t}\|\cdot\|}\right)(y_0)\leq\tau_x\left(e^{-\frac{c}{4t}\left(\sqrt{\|\cdot\|}+d(y,y_0)\right)^2}\right)(y_0),\quad \tau_x\left(e^{-\frac{c}{t}\|\cdot\|}\right)(y)\leq\tau_x\left(e^{-\frac{c}{4t}\left(\sqrt{\|\cdot\|}+d(y,y_0)\right)^2}\right)(y).$$

Therefore, from (4.4) and (4.7),

$$\begin{aligned}
&\int_{d_G(x,y)>2d(y,y_0)}\left|K^{(1)}(x,y)-K^{(1)}(x,\ y_0)\right|\vartheta_{k,1}(x)\,dx\\
&\leq Cd(y,y_0)\int_0^1\frac{1}{t^{2\langle k\rangle+N+\frac{1}{2}}}\left(\int_{\mathbb{R}^N}\tau_{y_0}\left(e^{-\frac{c}{4t}\left(\sqrt{\|\cdot\|}+d(y,y_0)\right)^2}\right)(x)\vartheta_{k,1}(x)\,dx\right.\\
&\quad\left.+\int_{\mathbb{R}^N}\tau_y\left(e^{-\frac{c}{4t}\left(\sqrt{\|\cdot\|}+d(y,y_0)\right)^2}\right)(x)\vartheta_{k,1}(x)\,dx\right)dt\\
&=Cd(y,y_0)\int_0^1\frac{1}{t^{2\langle k\rangle+N+\frac{1}{2}}}dt\int_{\mathbb{R}^N}2e^{-\frac{c}{4t}\left(\sqrt{\|x\|}+d(y,y_0)\right)^2}\vartheta_{k,1}(x)\,dx\\
&\leq Cd(y,y_0)\int_0^\infty r^{2\langle k\rangle+N-2}dr\int_0^1\frac{2}{t^{2\langle k\rangle+N+\frac{1}{2}}}e^{-\frac{c}{4t}\left(\sqrt{r}+d(y,y_0)\right)^2}dt\\
&\leq Cd(y,y_0)\int_0^\infty\frac{r^{2\langle k\rangle+N-2}}{\left(\sqrt{r}+d(y,y_0)\right)^{2\left(2\langle k\rangle+N-\frac{1}{2}\right)}}dr\int_0^\infty\frac{2}{u^{2\langle k\rangle+N+\frac{1}{2}}}e^{-\frac{c}{4u}}du\\
&\leq Cd(y,y_0)\int_0^\infty\frac{1}{\left(\sqrt{r}+d(y,y_0)\right)^3}dr=C.
\end{aligned}$$

The proof of Theorem 6.1 is complete for $a=1$. □

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