Small Sample Properties of OLS and GLS for the Regression Model with I(d) Regressor and Short Memory Error Term

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Abstract

We investigate the small sample properties of OLS and GLS for the regression model with I(d) regressor ($d \geqq 0$) and short memory error term by Monte Carlo simulation. We show how small sample properties vary depending on d, ratio of variances, short memory parameters and sample sizes.

1. Introduction

In the present paper, we study the asymptotic properties of OLS and GLS for the regression model with I(d) regressor ($d \geqq 0$) and short memory error term, and investigate the small sample properties of these estimators by Monte Carlo simulation.

In section 2, the model and assumptions are given. In section 3, we investigate the asymptotic properties of OLS and GLS for three cases of I(d) regressor (i.e. d=0, 0<d<1/2, d>1/2). In section 4, small sample properties of OLS and GLS are investigated by Monte Carlo simulation. A brief summary is given in section 5.

2. The Model and Assumptions

Let us consider the following simple stochastic regression model,

$y_t = \beta x_t + u_t, \quad t = 1, 2, \dots, T,$

where the independent variable x_t and the error term u_t follow certain stochastic processes described below and β is an unknown parameter to be estimated.

We set the following assumptions 1～3, where L is a lag operator.

Assumption 1

(1) $(1-L)^d x_t = w_t = \psi(L)\varepsilon_t,\ \psi(L) = \sum_{i=0}^{\infty} \psi_i L^i\ (\psi_0 = 1),\ \sum_{i=0}^{\infty} i\ |\psi_i| < \infty,$

and all roots of $\psi(z) = 0$ are outside the unit circle.

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Assumption 2

(2) $\phi(L)u_t = v_t,\ \phi(L) = \sum_{i=0}^{p} \phi_i L_i \ (\phi_0 = 1),$

and all roots of $\phi(z) = 0$ are outside the unit circle.

Assumption 3

$$(\varepsilon_t, v_t)' \sim I.I.D.((0, 0)', Diag(\sigma_\varepsilon^2, \sigma_v^2)).$$

In particular, in Assumption 1 we assume that x_t follows an I(d) process (Integrated process of order d). Specifically, x_t follows a short memory (stationary) process or a long memory (stationary) process depending on d=0 or 0<d<1/2. If d>1/2, x_t follows a nonstationary process. In the present paper we assume x_t=0. We exclude the case when d=1/2 (i.e., just nonstationary case), which needs different treatments. In Assumption 2, we assume that u_t follows a short memory (stationary) AR(p) process, the spectral density function of which is given by

(3) $f_u(\lambda) = \frac{\sigma_v^2}{2\pi} \frac{1}{|\phi(e^{i\lambda})|^2}, \quad (-\pi < \lambda \leqq \pi).$

The model can be written in conventional vector form as

$$y = x\beta + u,$$

where y, x, and u are given as follows.

(4) $y = (y_1, y_2, \ldots, y_T)', \quad x = (x_1, x_2, \ldots, x_T)', \quad u = (u_1, u_2, \ldots, u_T)'.$

3. Asymptotic Properties of OLS and GLS for Three Cases of I(d) Regressor

We introduce two estimators of β, that is, OLS (Ordinary Least Squares Estimator) $\hat{\beta}_0$ and GLS (Generalized Least Squares Estimator) $\hat{\beta}_G$ defined respectively as follows.

(5) $\hat{\beta}_0 = \frac{x'y}{x'x},$

(6) $\hat{\beta}_G = \frac{x'\Sigma^{-1}y}{x'\Sigma^{-1}x},$

where Σ is the covariance matrix of u given by

(7) $$\Sigma = E(uu') = \frac{\sigma_v^2}{1-\phi^2} \begin{bmatrix} 1 & \phi & \cdots & \phi^{T-1} \\ \phi & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \phi \\ \phi^{T-1} & \cdots & \phi & 1 \end{bmatrix}.$$

In the present paper, we consider the statistical properties of OLS $\hat{\beta}_0$ and GLS $\hat{\beta}_G$ for the following three cases.

Case 1 (d=0)

x_t is a short memory (stationary) process defined by

(8) $x_t = w_t = \psi(L)\varepsilon_t$.

x_t has a spectral density function

(9) $f_x(\lambda) = \dfrac{\sigma_\varepsilon^2}{2\pi} \mid \psi(e^{i\lambda}) \mid^2, \ (-\pi < \lambda \leqq \pi)$.

Case 2 (0<d<1/2)

x_t is a long memory (stationary) process defined by

(10) $(1-L)^d x_t = w_t = \psi(L)\varepsilon_t$, $(0<d<1/2)$.

x_t has a spectral density function

(11) $f_x(\lambda) = \dfrac{\sigma_\varepsilon^2}{2\pi} \mid \psi(e^{i\lambda}) \mid^2 \dfrac{1}{\mid 1 - e^{i\lambda} \mid^{2d}}, \ (-\pi < \lambda \leqq \pi)$.

Case 3 (d>1/2)

x_t is a nonstationary process defined by

(12) $(1-L)^d x_t = w_t = \psi(L)\varepsilon_t$, $(d>1/2)$.

We have the following Theorem for three cases described above.

Theorem

(1) Case 1 (d=0)

(13) $T^{\frac{1}{2}}(\hat{\beta}_0 - \beta) \longrightarrow N(0, V_1)$,

(14) $T^{\frac{1}{2}}(\hat{\beta}_G - \beta) \longrightarrow N(0, V_2)$,

where

(15) $V_1 = \dfrac{1}{(V(x_t))^2}\left(2\pi \int_{-\pi}^{\pi} f_x(\lambda) f_u(\lambda) d\lambda\right)$,

(16) $V_2 = \left(\dfrac{1}{2\pi} \int_{-\pi}^{\pi} f_x(\lambda) f_u^{-1}(\lambda) d\lambda\right)^{-1}$.

(2) Case 2 (0<d<1/2)

Same as Case 1.

(3) Case 3 (d>1/2).

(17) $T^d(\hat{\beta}_0 - \beta)$ *and* $T^d(\hat{\beta}_G - \beta) \longrightarrow MN(0, V_3)$,

where

(18) $V_3 = \frac{\sigma_v^2}{\phi^2(1)} \frac{1}{\int_0^1 F_{d-1}^2(r)dr}$,

(19) $F_{d-1}(r) = \frac{1}{\Gamma(d)} \psi(1)\sigma_\varepsilon \int_0^1 (r-s)^{d-1} dW_\varepsilon(s)$,

and $W_\varepsilon(s)$ is a standard Brownian motion defined on the probability space (Ω, F, P).

Case 1 and 2 were proved by Robinson and Hidalgo(1997) and Choy and Taniguchi(1999), for example. Case 3 was proved by Sugihara(1997).

Note that in Case 1 and 2($0 \leqq d < 1/2$), probability orders of OLS and GLS are $O_p\left(\frac{1}{T^{\frac{1}{2}}}\right)$ and asymptotic distributions are normal with $V_1 \geqq V_2$. On the other hand, in Case 3 ($d > 1/2$), OLS and GLS have the same asymptotic mixed normal distribution, and their probability orders are $O_p\left(\frac{1}{T^d}\right)$.

4. Small Sample Properties of OLS and GLS for Three Cases of I(d) Regressor

Using the following simple regression model,

$y_t = \beta x_t + u_t$,

$\phi(L)u_t = v_t$, $\phi(L) = 1 - \phi L$, $|\phi| < 1$,

we conduct Monte Carlo simulation to find the small sample properties of OLS and GLS for the three cases described below.

Case 1 (d=0) $x_t = \varepsilon_t$.

Case 2 ($0 < d < 1/2$) $(1-L)^d x_t = \varepsilon_t$.

Case 3 ($d > 1/2$) $(1-L)^d x_t = \varepsilon_t$.

For each combination of d=0, 0.4, 1.0, 1.4, T=50, 100, ϕ=–0.9, 0.3, 0.9 and $K \equiv \sigma_v^2/\sigma_\varepsilon^2$ = 1/10, 1, basic statistics, empirical cumulative distribution functions are computed using the results of 30,000 replications of Monte Carlo simulation.

The basic statistics of the simulation results are given in Tables 1 and 2. The empirical distribution functions for T=100 are given in Fig. 1～8.

From the Theorem (Case 1), we find that for d=0,

(20) $V_1 = \frac{\sigma_v^2}{\sigma_\varepsilon^2} \frac{1}{1-\phi^2}$,

(21) $V_2 = \frac{\sigma_v^2}{\sigma_\varepsilon^2} \frac{1}{1+\phi^2}$,

Table 1 Basic Statistics

d	T	ϕ	$\sigma_v^2/\sigma_\varepsilon^2$	ESTIMATOR	MEAN	VARIANCE	SD	skewness	kurtosis
0.0	50	0.9	1	GLS	–0.00066	0.60	0.7734	–0.03157	3.2458
		0.9	1	OLS	–0.00633	5.47	2.3389	–0.04722	4.0554
		0.3	1	GLS	–0.00178	0.96	0.9817	0.01739	3.1052
		0.3	1	OLS	–0.00183	1.14	1.0694	0.01756	3.1075
		–0.9	1	GLS	–0.00265	0.60	0.7765	0.00998	3.2037
		–0.9	1	OLS	–0.01035	5.49	2.3428	–0.00586	4.0813
0.0	50	0.9	1/10	GLS	–0.00069	0.06	0.2440	–0.01460	3.2415
		0.9	1/10	OLS	–0.00147	0.55	0.7383	–0.02620	4.1796
		0.3	1/10	GLS	–0.00089	0.10	0.3107	–0.01706	3.1275
		0.3	1/10	OLS	–0.00093	0.11	0.3384	–0.01741	3.1278
		–0.9	1/10	GLS	0.00400	0.06	0.2441	–0.00207	3.2586
		–0.9	1/10	OLS	0.01073	0.54	0.7355	–0.01470	4.0978
0.4	50	0.9	1	GLS	–0.00274	0.85	0.9242	0.03905	3.1576
		0.9	1	OLS	–0.00226	15.90	3.9878	0.05057	3.4845
		0.3	1	GLS	0.00635	0.92	0.9611	0.03285	3.2315
		0.3	1	OLS	0.00701	1.10	1.0493	0.03459	3.2494
		–0.9	1	GLS	0.00079	0.30	0.5458	–0.00953	3.4677
		–0.9	1	OLS	0.00109	2.00	1.4125	–0.02575	5.0278
0.4	50	0.9	1/10	GLS	–0.00049	0.09	0.2944	–0.01707	3.1893
		0.9	1/10	OLS	–0.00239	1.62	1.2728	–0.02478	3.5178
		0.3	1/10	GLS	0.00276	0.09	0.3043	–0.00743	3.1464
		0.3	1/10	OLS	0.00299	0.11	0.3322	–0.00713	3.1540
		–0.9	1/10	GLS	–0.00002	0.03	0.1731	–0.01353	3.6276
		–0.9	1/10	OLS	–0.00047	0.20	0.4480	–0.05107	5.1191
1.0	50	0.9	1	GLS	0.04543	45.64	6.7558	–0.00290	3.3140
		0.9	1	OLS	0.10077	248.42	15.7612	0.01212	4.5953
		0.3	1	GLS	–0.00544	9.80	3.1302	–0.01959	6.2659
		0.3	1	OLS	–0.00554	10.30	3.2097	–0.02239	6.5308
		–0.9	1	GLS	0.00850	1.66	1.2871	0.01354	7.5572
		–0.9	1	OLS	0.01400	3.55	1.8853	–0.18668	19.5525
1.0	50	0.9	1/10	GLS	–0.01509	4.53	2.1273	–0.01117	3.2794
		0.9	1/10	OLS	–0.03551	24.72	4.9721	–0.03307	4.6904
		0.3	1/10	GLS	–0.00154	0.98	0.9878	–0.01479	5.5220
		0.3	1/10	OLS	–0.00144	1.02	1.0122	–0.01379	5.7101
		–0.9	1/10	GLS	0.00026	0.16	0.4037	–0.10073	7.4193
		–0.9	1/10	OLS	0.00012	0.35	0.5893	–0.12580	13.5696
1.4	50	0.9	1	GLS	0.10061	427.52	20.6766	0.03019	4.9008
		0.9	1	OLS	0.18478	1055.64	32.4906	0.10704	9.4882
		0.3	1	GLS	0.00014	39.34	6.2721	–0.01009	11.5894
		0.3	1	OLS	–0.00013	40.01	6.3256	–0.00471	11.9241
		–0.9	1	GLS	0.00660	5.94	2.4377	–0.04686	11.9944
		–0.9	1	OLS	0.00986	8.42	2.9020	–0.05155	19.8936
1.4	50	0.9	1/10	GLS	–0.04056	42.30	6.5037	0.00004	4.8350
		0.9	1/10	OLS	–0.05724	103.41	10.1690	0.10475	8.8753
		0.3	1/10	GLS	–0.01130	3.88	1.9708	–0.08169	13.8532
		0.3	1/10	OLS	–0.01129	3.95	1.9882	–0.10825	14.8007
		–0.9	1/10	GLS	–0.00544	0.60	0.7729	–0.04407	11.7718
		–0.9	1/10	OLS	–0.00566	0.84	0.9154	–0.02387	16.0956

Table 2 Basic Statistics

d	T	ϕ	$\sigma_v^2/\sigma_\varepsilon^2$	ESTIMATOR	MEAN	VARIANCE	SD	skewness	kurtosis
0.0	100	0.9	1	GLS	−0.00509	0.57	0.7577	0.00329	3.1177
		0.9	1	OLS	−0.01672	5.38	2.3190	0.00705	3.5428
		0.3	1	GLS	0.00307	0.95	0.9753	0.03609	3.0349
		0.3	1	OLS	0.00335	1.13	1.0649	0.03643	3.0367
		−0.9	1	GLS	−0.00356	0.57	0.7562	0.00291	3.0487
		−0.9	1	OLS	−0.00918	5.35	2.3131	0.01869	3.5504
0.0	100	0.9	1/10	GLS	−0.00079	0.06	0.2394	−0.01042	3.0288
		0.9	1/10	OLS	−0.00199	0.54	0.7339	−0.00537	3.4965
		0.3	1/10	GLS	−0.00191	0.09	0.3073	−0.00290	3.0552
		0.3	1/10	OLS	−0.00214	0.11	0.3355	−0.00375	3.0560
		−0.9	1/10	GLS	−0.00163	0.06	0.2404	−0.02366	3.1523
		−0.9	1/10	OLS	−0.00540	0.54	0.7339	−0.03775	3.6239
0.4	100	0.9	1	GLS	0.00731	0.82	0.9071	−0.00167	3.0855
		0.9	1	OLS	0.03815	18.10	4.2546	0.00710	3.2642
		0.3	1	GLS	−0.00321	0.88	0.9403	0.00382	3.0880
		0.3	1	OLS	−0.00364	1.06	1.0279	0.00327	3.0989
		−0.9	1	GLS	0.00354	0.26	0.5104	−0.03178	3.2816
		−0.9	1	OLS	0.00819	1.67	1.2921	−0.05460	4.1651
0.4	100	0.9	1/10	GLS	0.00390	0.08	0.2884	−0.00236	3.0770
		0.9	1/10	OLS	0.01853	1.83	1.3510	−0.00561	3.2388
		0.3	1/10	GLS	−0.00238	0.09	0.2971	0.01320	3.0921
		0.3	1/10	OLS	−0.00258	0.11	0.3248	0.01385	3.1036
		−0.9	1/10	GLS	−0.00098	0.03	0.1606	0.02539	3.3108
		−0.9	1/10	OLS	−0.00169	0.16	0.4054	0.03809	4.0236
1.0	100	0.9	1	GLS	0.03523	77.24	8.7887	0.02308	3.3528
		0.9	1	OLS	0.06326	335.06	18.3046	0.02569	4.9165
		0.3	1	GLS	0.00727	10.52	3.2438	0.00047	5.9631
		0.3	1	OLS	0.00736	10.84	3.2922	0.00023	6.1272
		−0.9	1	GLS	−0.00430	1.60	1.2648	−0.03746	6.9580
		−0.9	1	OLS	−0.00640	2.58	1.6051	−0.06598	11.6128
1.0	100	0.9	1/10	GLS	0.00128	7.88	2.8064	0.00091	3.2803
		0.9	1/10	OLS	−0.00968	34.49	5.8727	−0.00874	4.7892
		0.3	1/10	GLS	0.00139	1.04	1.0219	−0.01513	5.9160
		0.3	1/10	OLS	0.00146	1.08	1.0368	−0.01070	6.0702
		−0.9	1/10	GLS	0.00118	0.16	0.3989	0.04722	6.8259
		−0.9	1/10	OLS	0.00107	0.25	0.5032	0.07168	10.2759
1.4	100	0.9	1	GLS	−0.05859	803.83	28.3519	−0.03443	6.0775
		0.9	1	OLS	−0.11005	1446.45	38.0322	−0.18234	12.0413
		0.3	1	GLS	0.01426	42.72	6.5362	−0.05495	12.8547
		0.3	1	OLS	0.01432	43.03	6.5596	−0.05671	13.0218
		−0.9	1	GLS	0.00876	5.95	2.4394	−0.25603	13.0724
		−0.9	1	OLS	0.00853	6.97	2.6396	−0.35006	15.5571
1.4	100	0.9	1/10	GLS	0.02580	82.10	9.0608	0.04102	5.8844
		0.9	1/10	OLS	0.03968	145.52	12.0630	0.07056	9.9178
		0.3	1/10	GLS	0.02635	4.23	2.0564	0.01887	15.6276
		0.3	1/10	OLS	0.02651	4.26	2.0642	0.02875	15.9756
		−0.9	1/10	GLS	0.00412	0.60	0.7732	0.09407	15.9310
		−0.9	1/10	OLS	0.00446	0.70	0.8393	0.10078	18.5168

Fig. 1 d=0.0, T=100, $\sigma_v^2/\sigma_\varepsilon^2=1$

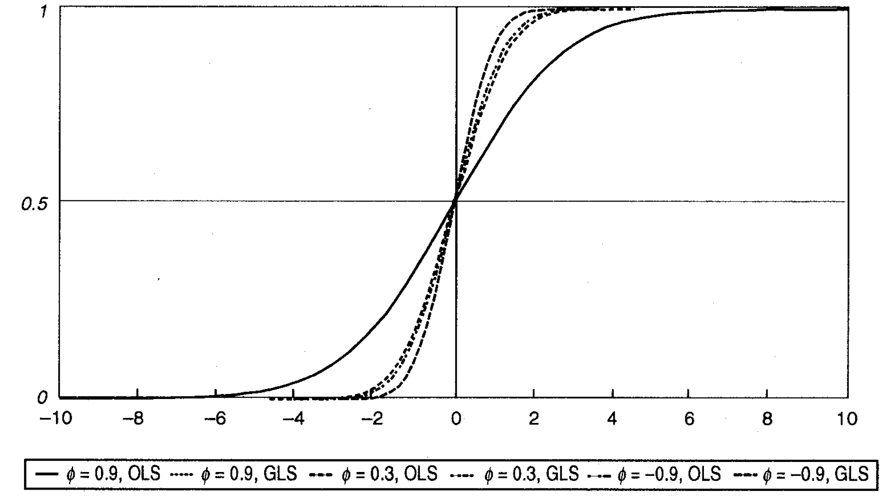


Fig. 2 d=0.0, T=100, $\sigma_v^2/\sigma_\varepsilon^2=1/10$

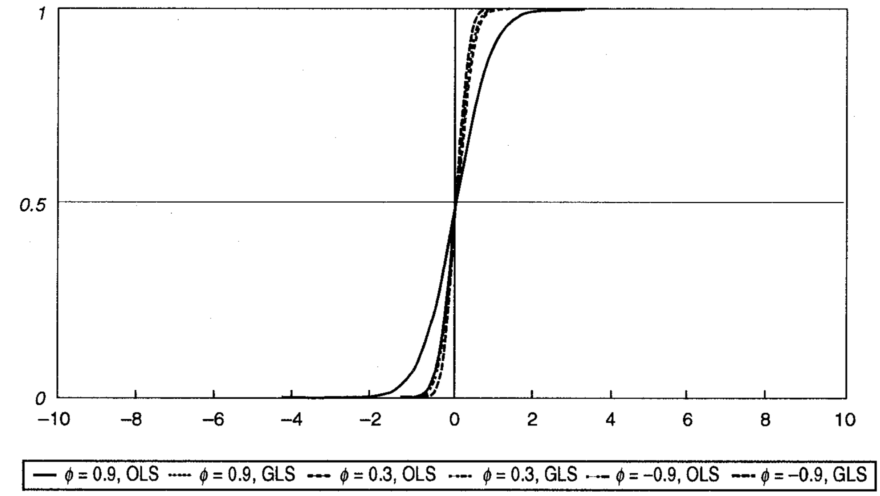


Fig. 3 d=0.4, T=100, $\sigma_v^2/\sigma_\varepsilon^2=1$

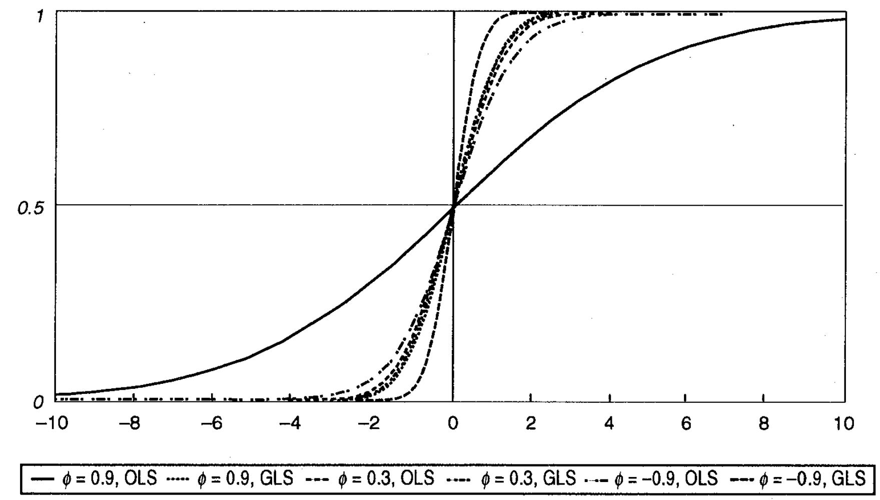


Fig. 4 d=0.4, T=100, $\sigma_v^2/\sigma_\varepsilon^2 = 1/10$

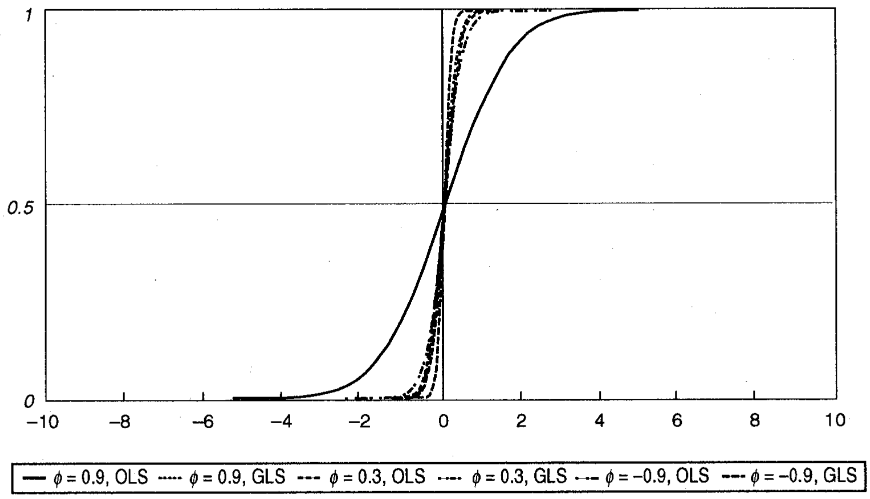


Fig. 5 d=1.0, T=100, $\sigma_v^2/\sigma_\varepsilon^2 = 1$

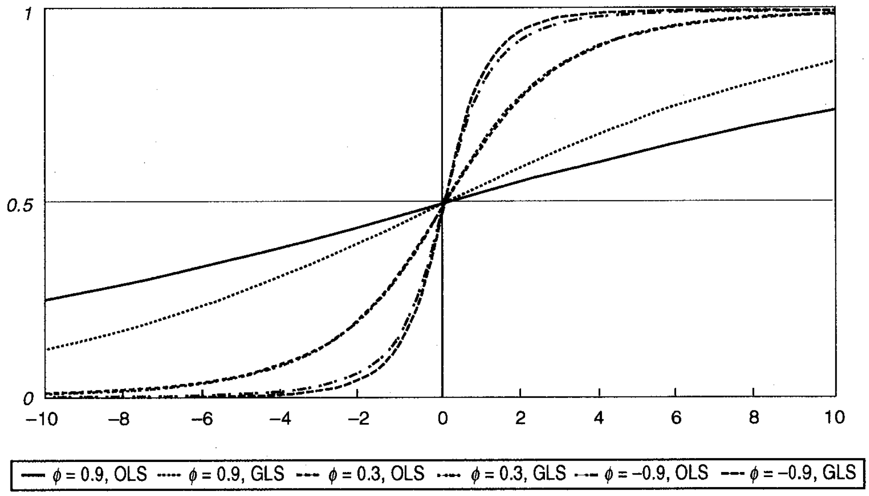


Fig. 6 d=1.0, T=100, $\sigma_v^2/\sigma_\varepsilon^2 = 1/10$

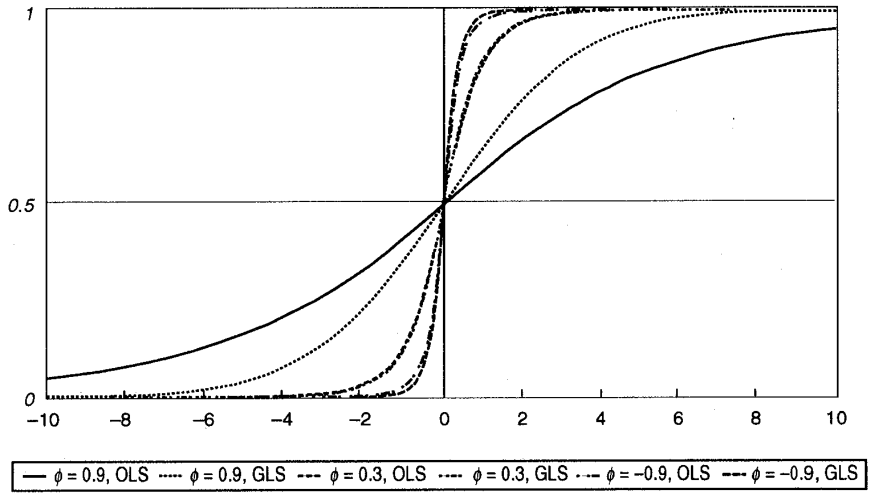


Fig. 7 d=1.4, T=100, $\sigma_v^2/\sigma_\varepsilon^2=1$

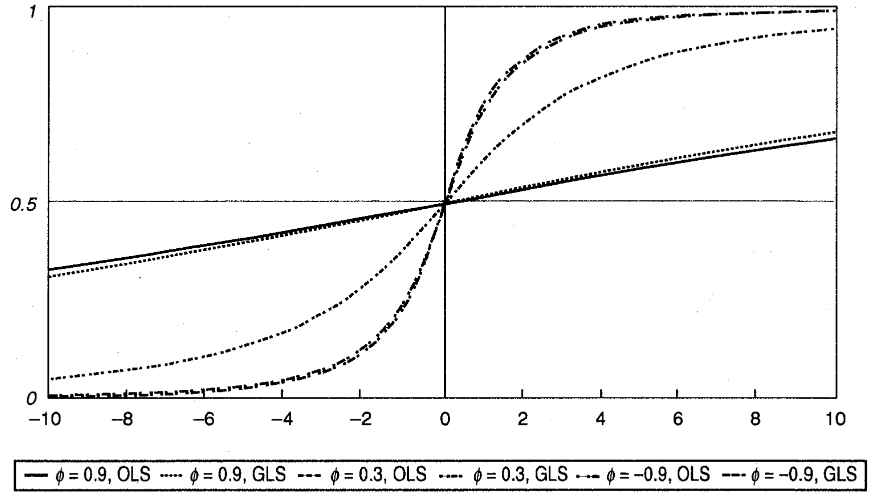
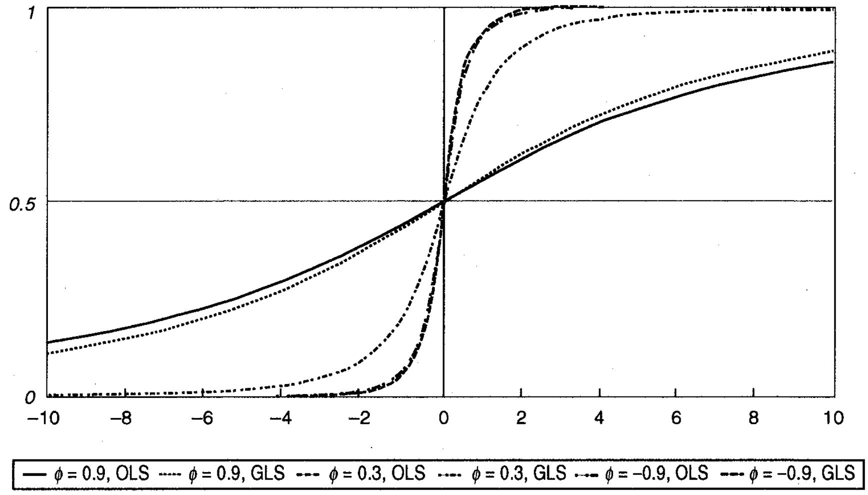


Fig. 8 d=1.4, T=100, $\sigma_v^2/\sigma_\varepsilon^2=1/10$



and $V_1 \geqq V_2$. From Fig. 1, 2 and Table 1, 2, we can see that this tendency becomes remarkable as K increases from 1/10 to 1 and ϕ increases from –0.9 to 0.9. Notice that in the short memory (stationary) case, in particular, OLS's of ϕ=0.9 and ϕ=–0.9 have the same asymptotic distribution, and this also holds true for the GLS's of ϕ=0.9 and ϕ=–0.9. As we can see from Fig. 1 and 2, kurtosis of OLS and GLS takes values around 3, reflecting the asymptotic normality of the estimators.

The same tendency also holds for the case of d=0.4, although the asymptotic distribution functions of OLS and GLS of ϕ=0.9 and ϕ=–0.9 are different.

In the case of $d=1.0$ and $d=1.4$, we find from the Theorem (Case 3) that OLS and GLS have the same asymptotic mixed normal distribution, and reflecting this asymptotic properties, all the figures of kurtosis in Table 1 and 2 take values larger than 3. This tendency becomes remarkable as d becomes large from 1 to 1.4 and K becomes large from 1/10 to 1. Although OLS and GLS have the same asymptotic mixed normal distribution, when $d=1$ and $T=100$ for example, we see from the Fig. 5 and 6 that difference between OLS and GLS becomes large when ϕ approaches to unity, for example $\phi=0.9$.

5. Summary

In the present paper, we investigate the small properties of OLS and GLS for the regression model with I(d) regressor ($d \geqq 0$) and short memory error term by Monte Carlo simulation. We show how small sample properties vary depending on d, $K \equiv \sigma_{\varepsilon}^2/\sigma_v^2$, ϕ, and T. Derivation of the asymptotic properties as well as small sample properties of the estimators for the long memory error term case is a task that remains for the future.

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