On the Detection of Common Factors and the Recovery of Raw Data for the Cointegrated Time Series

Soichi SUGIHARA*

Abstract

We consider a statistical method to detect the common factors of the cointegrated time series and to recover the raw data from these common factors. We apply this method to the Brazilian / US currency official and black market exchange rate data.

I Introduction

There has been an increasing interest in the nature of relationship among non-stationary multiple time series during the last two decades. Such terms as unit root test, cointegration relationship, error correction representation and common factors represent the new ideas of this rapidly developing area. In the present paper we investigate a statistical method to detect the "common factors" ("common trends") and to recover the raw data for the cointegrated time series.

This paper is organized as follows: In section II, we review several theoretical methods of cointegration. In section III, we investigate a statistical method to detect the common factors of the cointegrated time series and to recover the raw data from these common factors. As an example, in section IV we analyze the empirical data of the Brazilian/US currency official and black market exchange rate given by Bessler and Yu[2]. Finally, section V provides brief summary of the paper.

II Cointegration and the model

Let x_t be an $(m \times 1)$ random vector which is at most I(1) and follows VAR(p) process with cointegration rank $r(0 \leq r \leq m)$,

$$\Phi(L)x_t = \mu + \varepsilon_t \qquad t = 1, 2, \ldots, T, \tag{1}$$

where $\Phi(L) = 1 - \sum_{i=1}^{p} \Phi_i L^i$ is a p-th order polynomial of lag operator L, and $rank\ (\Phi(1)) = r$. $|\Phi(z)| = 0$ has (m - r) unit roots. $\Phi(1) = 0$ if x_t is I(1) and $r = 0$, and $rank(\Phi(1)) = m$ if x_t is I(0). μ is an $(m \times 1)$ constant vector and in the present paper we assume ε_t is I.I.D$(0, \Sigma)$.

* Professor, School of Business Administration, Kwansei Gakuin University

When $rank(\Phi(1)) = r\ (0 < r < m)$, there exist $(m \times r)$ matrices A and B with $rank(A) = rank(B) = r$ and $-\Phi(1) = AB'$.

We define $\Gamma(L) = 1 - \sum_{i=1}^{p-1} \Gamma_i L^i$ through the identity

$$\Phi(L) = \Phi(1)L + (1-L)\Gamma(L) \qquad (\Gamma_i = -\sum_{j=i+1}^{p} \Phi_j,\ i = 1, 2, \ldots, p-1) \tag{2}$$

so that x_t has an error correction representation

$$\Delta x_t = \sum_{i=1}^{p-1} \Gamma_i \Delta x_{t-i} + AB' x_{t-1} + \mu + \varepsilon_t, \tag{3}$$

where $\Delta \equiv 1 - L$.

Eq.(1) has another equivalent expression

$$\Delta x_t = \Theta(L)(\mu + \varepsilon_t) \tag{4}$$

where $\Theta(L) = 1 + \sum_{i=1}^{\infty} \Theta_i L^i$ and $rank(\Theta(1)) = m - r$. $\Theta(1)$ can be expressed by

$$\Theta(1) = B_{\perp}(A_{\perp}' \Gamma(1) B_{\perp})^{-1} A_{\perp}' \tag{5}$$

where $A_{\perp}$, $B_{\perp}$ are both $m \times (m - r)$ with $rank(A_{\perp}) = rank(B_{\perp}) = m - r$ and $A_{\perp}'A = 0$, $B_{\perp}'B = 0$.

We define $\Theta^*(L) = \sum_{i=0}^{\infty} \Theta_i^* L^i \quad (\Theta_i^* = -\sum_{j=i+1}^{\infty} \Theta_j \quad i = 0, 1, \ldots)$ through the identity

$$\Theta(L) = \Theta(1) + (1-L)\Theta^*(L)\,. \tag{6}$$

Using a multivariate version of Beveridge-Nelson decomposition, x_t can be decomposed as

$$\begin{aligned} x_t &= \sum_{s=1}^{t} \Delta x_s + x_0 \qquad (7) \\ &= \Theta(1)(\mu t + \sum_{s=1}^{t} \varepsilon_s) + \Theta^*(L)\varepsilon_t + (x_0 - \Theta^*(L)\varepsilon_0)\,. \end{aligned}$$

Here, $\Theta(1)\mu t$ and $\Theta(1)\sum_{s=1}^{t} \varepsilon_s$ constitute the deterministic and stochastic trend of long run components of x_t respectively, $\Theta^*(L)\varepsilon_t$ is the short run components of x_t and $(x_0 - \Theta^*(L)\varepsilon_0)$ is the initial value.

Since μ can be expressed as

$$\mu = A\beta_0 + A_{\perp}\alpha_0\,, \tag{8}$$

eq. (3) and (7) can also be expressed as

$$\Delta x_t = \sum_{i=1}^{p-1} \Gamma_i \Delta x_{t-i} + AB^{*\prime} x_{t-1}^{*} + A_{\perp}\alpha_0 + \varepsilon_t \tag{9}$$

where $B^* = (B', \beta_0)'$, $x_{t-1}^* = (x_{t-1}', 1)'$ and

$$x_t = \Theta(1)(A_\perp \alpha_0 t + \sum_{s=1}^{t} \varepsilon_s) + \Theta^*(L)\varepsilon_t + (x_0 - \Theta^*(L)\varepsilon_0) . \tag{10}$$

Hereafter when $\alpha_0 = 0$ in particular, we call this case restricted while when $\alpha_0 \neq 0$, we call this case unrestricted. In the restricted case, cointegration matrix B^* has a vector β_0 and in particular x_t does not have a deterministic trend. The restriction can be tested using Johansen's procedure. (see Johansen[6].)

III Detection of common factors and recovery of raw data

Let H_1 and H_2 be $H_1 = A(A'A)^{-\frac{1}{2}}$, $H_2 = A_\perp (A_\perp' A_\perp)^{-\frac{1}{2}}$ and $H = (H_1, H_2)$. Using $HH' = I_m$, Stock and Watson[7] define "common trend" of x_t as

$$f_t = H_2' (\mu t + \sum_{s=1}^{t} \varepsilon_s) . \tag{11}$$

The long run component can be represented as $1_t = \Theta(1)H_2 f_t$.

Whether restricted or unrestricted, using eq.(3), we can show that their common trend can be detected from the data as

$$\begin{aligned} f_t &= H_2' \sum_{s=1}^{t} \Gamma(L)\Delta x_s \\ &= H_2' \Gamma(L)(x_t - x_0) . \end{aligned} \tag{12}$$

In the restricted case, the common trend is constructed only by the stochastic trend $H_2'\sum_{s=1}^{t} \varepsilon_s$, while in the unrestricted case, the common trend is constructed by deterministic trend with nonzero slope $H_2'A_\perp\alpha_0$ as well as stochastic trend $H_2'\sum_{s=1}^{t} \varepsilon_s$. If we want to recover x_t using the common trend f_t, one way to do so is by $\tilde{\Theta}(1)\tilde{H}_2\tilde{f}_t + x_0$, where the notation "~" indicates corresponding estimator. Note that this depends directly on the initial value x_0.

On the other hand, recently Gonzalo and Granger[3] proposed the following "permanent-transitory decomposition" (P–T decomposition) of x_t using the identity $I_m = B_\perp(A_\perp'B_\perp)^{-1}A_\perp' + A(B'A)^{-1}B'$ as.

$$x_t = A_1 f_t + A_2 z_t , \tag{13}$$

where $f_t = A_\perp' x_t$, $z_t = B'x_t$ and $A_1 = B_\perp(A_\perp'B_\perp)^{-1}$, $A_2 = A(B'A)^{-1}$.

They define f_t as the "common factors" of x_t. Although their original model does not contain a constant vector μ, its inclusion can be treated in the same way as theirs.

Note that in the restricted case,

$$A_2 E(z_t) = -A_2\beta_0 \equiv C_0 , \tag{14}$$

while in the unrestricted case

$$A_2E(z_t) = A_2\,\{-\beta_0 + (A'A)^{-1}(A'\,\Gamma(1)B_\perp)\,(A_\perp{}'\,\Gamma(1)B_\perp)^{-1}\,A_\perp{}'A_\perp\,\alpha_0\} \tag{15}$$
$$\equiv C_1\ .$$

Therefore if we want to recover x_t using the common factor f_t, one way to do so is by $\widetilde{A}_1\widetilde{f}_t + \widetilde{C}$ where $\widetilde{C}$ is $\widetilde{C}_0$ or $\widetilde{C}_1$ as the case may be. The other way might be to replace C with A_2z_0, but this depends directly on the initial value x_0.

IV Applications

Using monthly data of the Brazilian / US currency official and black market exchange rates (1973 through 1989), Bessler and Yu[2] has found the cointegration relationship between these log log transformed time series. In the present paper, we want to detect the common factor of these log log transformed time series and to recover the raw data using our method described in the previous section.

Let x_{1t} and x_{2t} be the log log transformed official and black market exchange rate respectively and $x_t = (x_{1t}, x_{2t})'$. Fig.1 and Fig.2 give plots of x_{1t} , x_{2t} and Δx_{1t} , Δx_{2t} respectively.

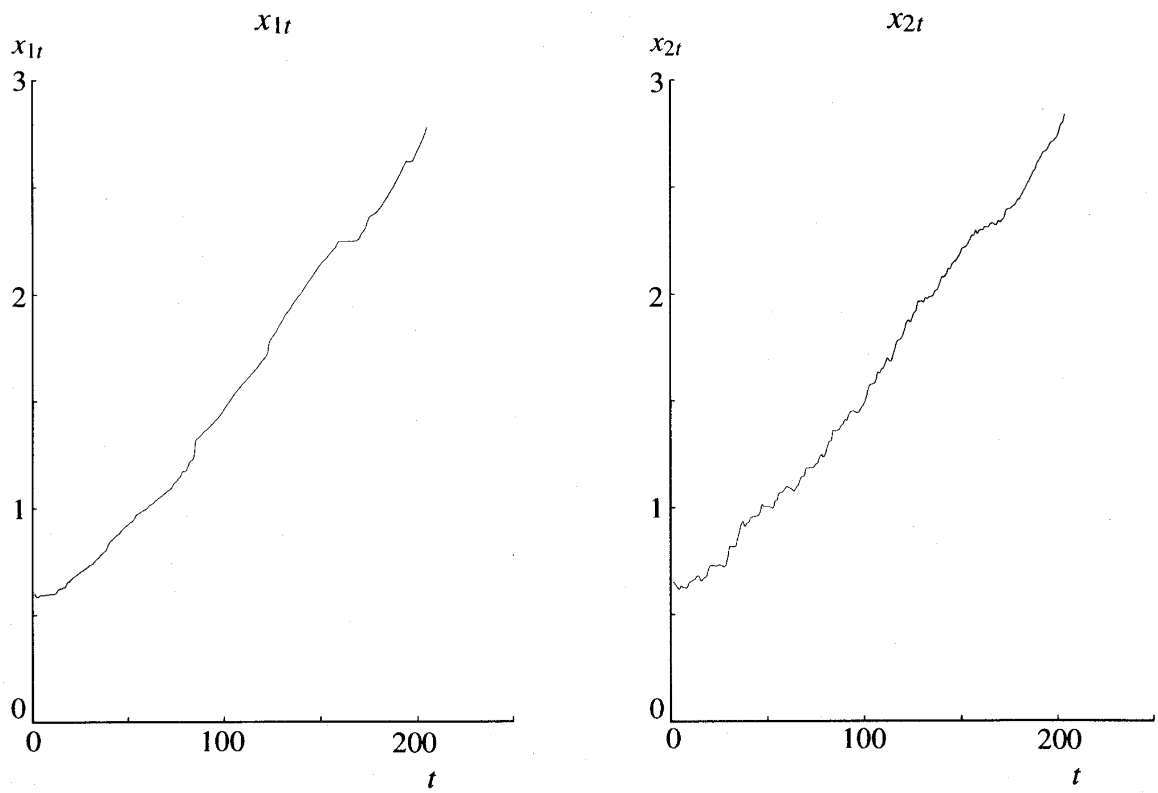


Fig. 1

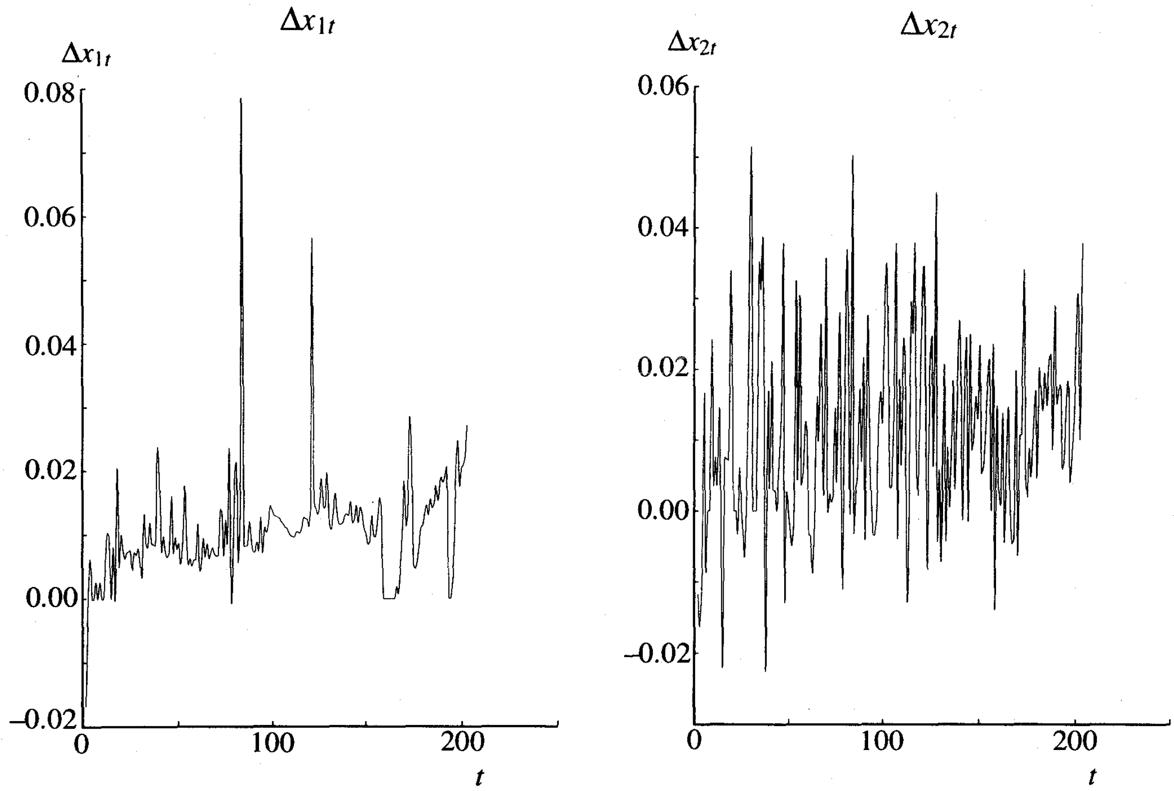


Fig. 2

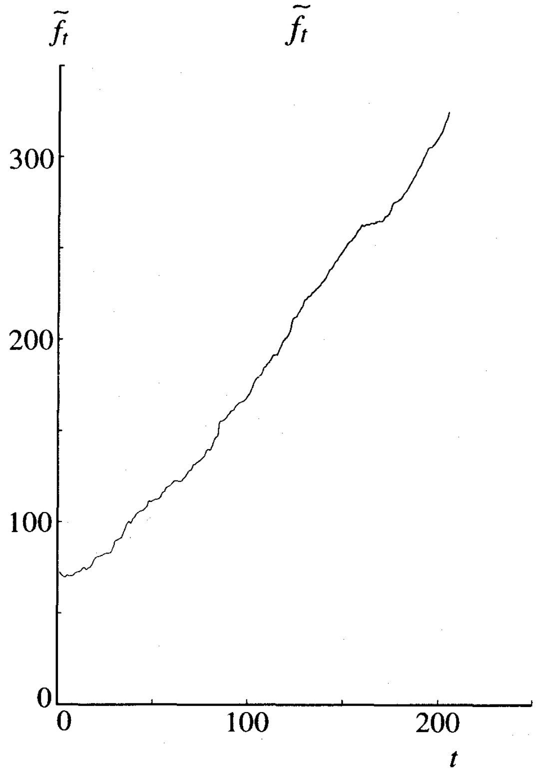
Bessler and Yu[2] has shown that the unrestricted model with $p = 2$ and $r = 1$ is an adequate model to analyze. Using their results, we estimate eq.(3) and get the following results.

$$\Delta x_t = \begin{pmatrix} 0.3459, & -0.0821 \\ 0.3270, & 0.0526 \end{pmatrix} \Delta x_{t-1} + \begin{pmatrix} -0.0737 \\ 0.0696 \end{pmatrix} (\,1, \;\; -1.0255\,)\, x_{t-1} + \begin{pmatrix} 0.0016 \\ 0.0130 \end{pmatrix} \tag{16}$$

((2,2) element of $\widetilde{\Gamma}_1$ is slightly different from Bessler and Yu[2].)

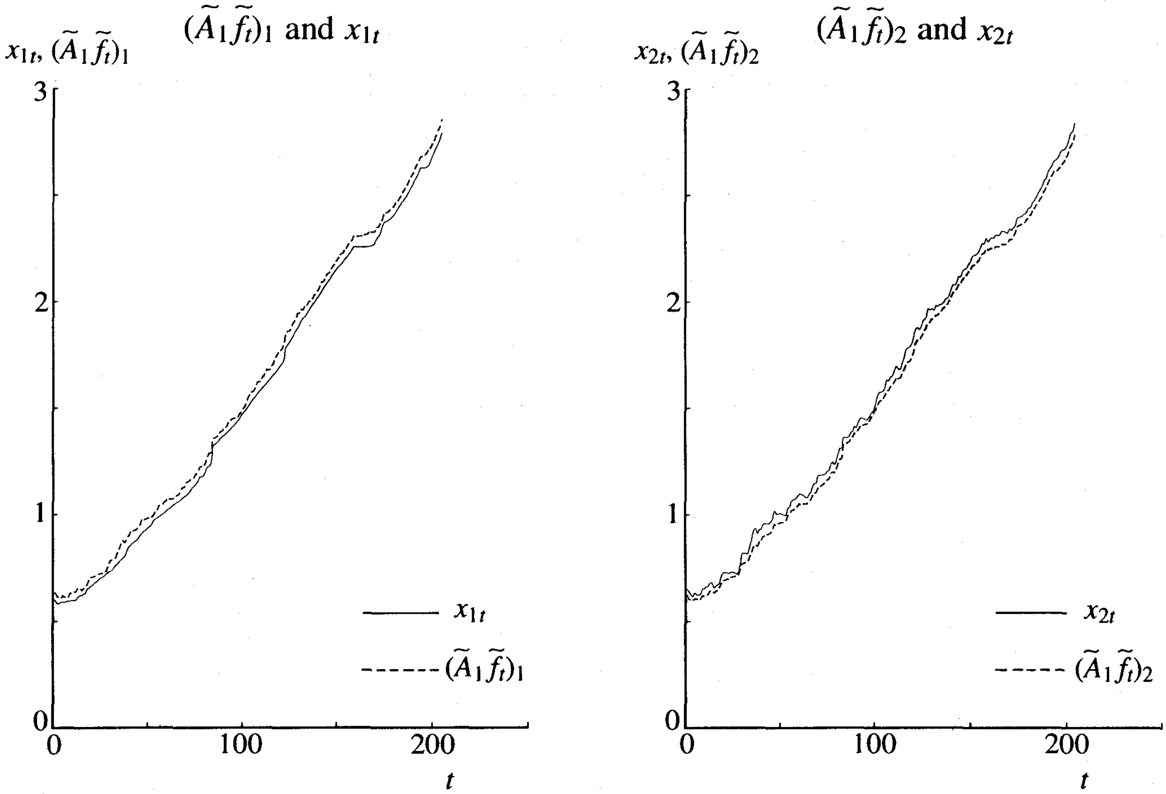
Common factor of x_t is detected from the data as $\widetilde{f}_t = \widetilde{A}_\perp{}' x_t$ and is plotted in Fig. 3. Permanent component $\widetilde{A}_1 \widetilde{f}_t$ is plotted in Fig. 4. x_t is recovered by $\widetilde{A}_1 \widetilde{f}_t + \widetilde{C}_1$ and is plotted in Fig. 5.

As we can see from Fig. 4 and 5, x_t is recoverd quite well by $\widetilde{A}_1 \widetilde{f}_t + \widetilde{C}_1$. $\widetilde{A}_1 \widetilde{f}_t + \widetilde{A}_2 \widetilde{Z}_0$ can also be used to recover x_t, but the differences between them are quite small and the graphs are similar to Fig. 5.



$\tilde{f}_t$
$\tilde{f}_t$
300
200
100
0
0
100
200
t

Fig. 3



$(\tilde{A}_1\tilde{f}_t)_1$ and x_{1t}
$x_{1t}, (\tilde{A}_1\tilde{f}_t)_1$
3
2
1
0
0
100
200
t
x_{1t}
$(\tilde{A}_1\tilde{f}_t)_1$
$(\tilde{A}_1\tilde{f}_t)_2$ and x_{2t}
$x_{2t}, (\tilde{A}_1\tilde{f}_t)_2$
x_{2t}
$(\tilde{A}_1\tilde{f}_t)_2$

Fig. 4

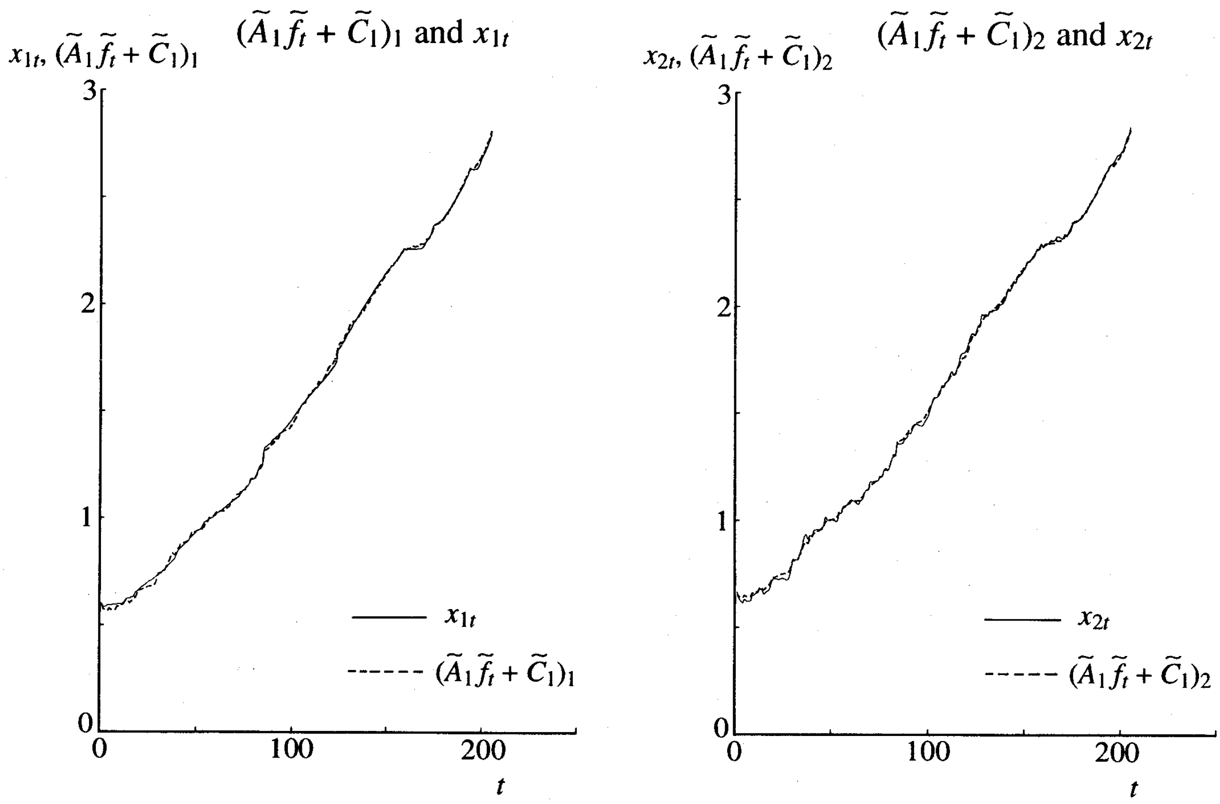


Fig. 5

V Summary

In the present paper, we investigated the statistical method to detect the common factors of the cointegrated time series with constant term and to recover the raw data from these common factors. We applied this method to the empirical data of the Brazilian/US currency official and black market exchange rate (Bessler and Yu[2]) and got reasonable results.

We are preparing another paper to apply our methods to cointegrated time series constructed from more than three variables.

References

[1] Anindya Banerjee, Juan J. Dolado, John W. Galbraith and David F. Hendry, (1993) *Co-integration, Error Correction, and the Econometric Analysis of Non-stationary Data.* Oxford University Press.

[2] David A. Bessler and Hui-Kuang Yu, (1994) "Official and black market exchange rates in Brazil : a case of the iterated logarithm," Applied Financial Economics, vol. 4, pp207-216.

[3] Gonzalo, Jesus and Clive W. J. Granger, (1995) "Estimation of common long-memory components in cointegrated systems," Journal of Business and Economic Statistics, vol. 13, pp27-35.

[4] Hatanaka, M., (1996) *Time Series Based Econometrics.* Oxford University Press.

[5] Johansen Søren, (1991) "Estimation and hypothesis testing of cointegration vectors in gaussian vector autoregressive model," Econometrica, vol. 59, pp1551-1580.

[6] Johansen Søren, (1995) *Likelihood-Based Inference in Cointegrated Vector Autoregressive Models.* Oxford University Press.

[7] Stock, James H. and Mark W. Watson, (1988) "Testing for common trends," Journal of the American Statistical Association, vol. 83, pp1097-1107.