Growth in a Stochastic Voracity Model

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This paper introduces uncertainty in the form of a geometric Brownian motion into a voracity model to seek some implications of uncertainty for growth in the feedback Nash equilibrium. We demonstrate that the voracity effect survives this extension, i.e., a technological progress is growth-reducing under the same condition made in the deterministic model.

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JEL : C73, O40

Keywords : stochastic dynamic game, geometric Brownian motion, feedback Nash equilibrium, voracity, growth

1 Introduction

The engine of economic growth is presumably one of the central concerns in economics. Given this fact, there is a so-called endogenous growth literature that theoretically and empirically identifies the determinant of the growth rate, providing a number of useful insights that are not addressed in the classical growth theory.[1)]

However, most of the endogenous growth models have overlooked the consequences of strategic interactions among self-interested agents by presuming a representative consumer. To our knowledge, Tornell and Velasco

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1) See, for example, Barro and Sala-i-Martin (2004) and Acemoglu (2008) for up-to-date textbooks of growth theory.

(1992) are the first to relax this assumption by combining an AK model with a differential game. One of their striking findings is that an increase in the productivity parameter is detrimental to growth since it induces each consumer to extract the resource stock faster and reduces aggregate saving. This implies that the conventional wisdom resting on the representative consumer assumption may not be useful in a seemingly more realistic case with strategic interactions.[2)]

The results of Tornell and Velasco (1992) are extended in a variety of directions, e.g., Long and Sorger (2006) and Mino (2006), but all of them commonly adopt a deterministic model. In view of the reality, nevertheless, the stock of fish and forests is subject to uncertain volatilities, and hence one needs a stochastic framework to model such uncertain situations. To this end, this paper constructs a stochastic dynamic game model of growth. While we follow Tornell and Velasco's (1992) modeling, the state variable evolves according to a geometric Brownian motion in our model. Although there are alternative ways to introduce uncertainty, our model has a advantage since it allows us to find the feedback (Markov perfect) Nash equilibrium in a closed form, i.e., the equilibrium strategy is explicitly solvable.[3)] We demonstrate that Tornell and Velasco's (1992) finding above is still valid even in our extended model. That is, their result holds regardless of the presence of uncertainty. After formally proving this, we discuss its intuition and implication.

This paper is organized as follows. Section 2 presents a model, and

2) Note that assuming a representative consumer is equivalent to assuming cooperation (collusion) among symmetric players. In this sense, the standard growth theory is viewed as a special case of a game-theoretic model with inter-player cooperation.

3) With the same motivation, Wang and Ewald (2010) construct a Fershtman-Nitzan (1991) model of public good provision that includes a geometric Brownian motion of the public good stock.

computes the feedback Nash equilibrium. Section 3 considers the impact of a technological progress and the degree of uncertainty on the growth rate in the feedback Nash equilibrium. Section 4 concludes.

2 A Model

We consider a stochastic version of the Tornell-Velasco (1992) model. There are $n \geq 1$ symmetric players that extract a renewable resource stock $x(t)$ over an infinite horizon.[4] Thus, the dynamic utility maximization problem of player i is formulated by

$$\max_{c_i} E\left(\int_0^\infty e^{-rt}\frac{c_i^{1-\theta}}{1-\theta}dt\right), \quad \theta \in (0,1)$$

$$\text{s.t. } dx = \left(ax - \sum_{j=1}^{n} c_j\right)dt + \sigma x dw, \tag{1}$$

where E is an expectation operator, c_i is consumption of player i, $r > 0$ is a discount rate, and dw is an increment of a Wiener process with σ denoting a standard error that measures the degree of uncertainty.[5]

Following the solution technique of Dockner et al. (2000) and Long (2010), let us construct a Hamilton-Jacobi-Bellman (HJB) equation of player i:

$$rV_i(x) = \max_{c_i}\left\{\frac{c_i^{1-\theta}}{1-\theta} + V_i'(x)\left[ax - c_i - (n-1)c(x)\right] + \frac{\sigma^2 x^2}{2}V_i''(x)\right\}, \tag{2}$$

where $V_i(x)$ is a value function, and $c(x)$ is a feedback (Markovian) strategy that is chosen by all the other players. The first-order condition is $c_i^{-\theta} = V_i(x)$, which is inverted to get the optimal consumption level $c(x) = \left[V'(x)\right]^{-1/\theta}$.[6] Substituting this into (2), and guessing the value function $V(x) = Ax^{1-\theta}/(1-\theta)$, we obtain

4) In what follows, the time argument t is suppressed unless any confusion arises.

5) Dixit and Pindyck (1994, Ch. 3) are a useful reference of the geometric Brownian motion.

6) Subscript i is dropped here since we focus on a symmetric equilibrium.

$$A = \left\{ \frac{2[r-(1-\theta)a]+\theta(1-\theta)\sigma^2}{2[1-n(1-\theta)]} \right\}^{-\theta}$$

$$c(x) = \frac{2[r-(1-\theta)a]+\theta(1-\theta)\sigma^2}{2[1-n(1-\theta)]} x. \tag{3}$$

To ensure the utility to be bounded, we make:[7)]

Assumption. $\left\{2[r-(1-\theta)a]+\theta(1-\theta)\sigma^2\right\}[1-n(1-\theta)] > 0.$

Eq. (3) gives the feedback strategy in the Nash equilibrium, which is made use of to investigate the role of uncertainty (σ in our context) in the next section.

3 Voracity and Growth

This section considers the effect of a technological progress (an increase in a) on the equilibrium growth rate. Because the feedback Nash equilibrium (3) is linear in x, the evolution of x follows $dx = gxdt + \sigma xdw$, where g is obtained as

$$g = a - n\frac{2[r-(1-\theta)a]+\theta(1-\theta)\sigma^2}{2[1-n(1-\theta)]} = \frac{2(a-nr)-n\theta(1-\theta)\sigma^2}{2[1-n(1-\theta)]}. \tag{4}$$

Thus, the expected value of the solution to the above differential equation becomes

$$E[x(t)] = x_0 e^{gt} = x_0 \exp\left\{ \frac{2(a-nr)-n\theta(1-\theta)\sigma^2}{2[1-n(1-\theta)]} t \right\}. \tag{5}$$

Eq. (5), which is a straightforward extension of Eq. (4b) in Tornell and Velasco (1992, p. 1213), leads to:

Proposition. *A technological progress (an increase in a) reduces the growth rate, and an increase in uncertainty (σ^2) raises the growth rate if and only if* $1-n(1-\theta) < 0$.

7) This assumption is just the same as Eq. (5) in Tornell and Velasco (1992, p. 1213).

Proof. Differentiating g with respect to a and σ^2 yields

$$\frac{\partial g}{\partial a} = \frac{1}{1-n(1-\theta)} < 0, \quad \frac{\partial g}{\partial \sigma^2} = -\frac{n\theta(1-\theta)}{2[1-n(1-\theta)]} > 0,$$

under $1-n(1-\theta) < 0$. ||

The intuition behind this result is as follows. When the production technology improves, i.e., a rises, stock accumulation is accelerated as a first effect. Each player optimally responds to such faster accumulation of the stock by increasing consumption. As a consequence of this collectively aggressive behavior, stock accumulation is impeded if the number of players is large enough to satisfy $1-n(1-\theta) < 0$ or equivalently $n > 1/(1-\theta)$, and hence growth becomes slower. If, on the contrary, the number of players is small so that $n < 1/(1-\theta)$, the first effect of increased a dominates the secondary effect of increasing consumption, which leads to higher growth.[8) In other words, when the negative effect caused by the tragedy of the commons is larger than the positive effect, saving and stock accumulation decrease, and we have lower growth. What is worth mentioning is that the uncertainty term σ^2 plays no role in this argument. In this sense, it is fair to say that the finding of Tornell and Velasco's (1992) deterministic model has a firm validity.

The effect of an increase in uncertainty $\left(\sigma^2\right)$ can be interpreted in a similar way. In a representative consumer model (the case with $n = 1$), increased uncertainty positively affects the consumption, and the resulting over-consumption reduces the growth rate. In contrast, if the number of consumers is sufficiently large, the more uncertain the change in the resource is, the less each player consumes. Because this last effect increases aggregate saving, growth is enhanced by an increase in uncertainty.

8) In the representative consumer model that is reproduced by setting $n = 1$, higher a necessarily raises the growth rate.

To sum, if the number of players is large, the effect of the tragedy of the commons plays a dominant role in the comparative statics outcomes, possibly reversing the results that are based on the representative consumer model.

4 Conclusion

We have reconsidered Tornell and Velasco's (1992) seminal work that demonstrates a growth-reducing possibility of a technological progress. It is revealed that their result is qualitatively valid even in a stochastic game model. Concretely, in the feedback Nash equilibrium, a technological improvement has a negative growth effect if the number of players is large enough.

Despite the above novelty, there admittedly remains much unexplored. First, we have extended the simplest version of the Tornell-Velasco (1992) model with one asset. The continuing validity of Tornell and Velasco's (1992) result may come from our adoption of the one-asset model. Tornell (1999), Tornell and Lane (1999), and Long and Sorger (2006), on the other hand, use a two-asset model to examine the role of insecure property rights for growth. Second, it may be possible to allow for a leader-follower structure of the game as is the main focus of Fujiwara (2012). Third and more importantly, there are other ways of introducing uncertainty. Dynamic games with a geometric Brownian motion are so tractable that they are widely used in the applications literature, e.g., Wirl (2007, 2008), but it is important to employ another specification of uncertainty with the same purpose as this paper.[9] All of these extensions are beyond the scope of this paper, but are worth trying as future research agenda.

9) See, for example, Dockner et al. (2000, Ch. 8).

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