〈研究ノート〉

A Detailed Derivation of the Distribution of Class Identification in a "Chance Society":

A Note on the Fararo-Kosaka Model[*]

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1 Purpose

Kenji Kosaka and Thomas J. Fararo (1991) proposed a model, hereafter referred to as "the Fararo-Kosaka model," to explain the generalization of images of stratification and distribution of class identification. Because the Fararo-Kosaka model is mathematically simple and has a number of meaningful derivations, the model has attracted many followers. In particular, in Japan, the Fararo-Kosaka model has been considered a plausible theoretical framework for explaining the middle classification phenomenon, which has been empirically observed in the last three decades.

This note presents a detailed derivation of the distribution of class identification in order to help learners, especially beginners in mathematical sociology, to gain a deeper understanding of the model.

2 Axioms of the Model

The complete axioms of the Fararo-Kosaka model are described in detail in Kosaka and Fararo (1991), Fararo and Kosaka (2003), and Kosaka (2006). Here, we briefly introduce the essence of the model axioms.

Axiom 1. There exists a multidimensional stratification system S that is composed of a Cartesian product of s characteristics C_1, C_2, $\cdots$, C_s and each C_i is ordered linearly. Each s-tuple representing a social class in $C_1 \times C_2 \times \cdots \times C_s$ is ordered lexicographically. In a word, S is mathematically defined as a linearly ordered set $(C_1 \times C_2 \times \cdots \times C_s, \leq_l)$ with a lexicographic order $\leq_l$.

We call each of C_1, C_2, $\cdots$, C_s a "dimension" and each component of C_i a "rank." In the following analysis, we assume that each dimension has an identical number of ranks more than 1 (i.e., the rank homogeneity assumption). The number of dimensions is denoted by s and the number of ranks in C_i is denoted by r. At this stage, we call S the "$s \times r$ stratification system." The ranks in C_i are denoted by integers from 0 as the lowest up to $r-1$ as the highest as follows:

$$C_i = \{0, 1, 2, \cdots, r-2, r-1\}.$$

An s-tuple representing a class in the stratification system is generally denoted by

$$(k_1, k_2, \cdots, k_s), \ k_i \in C_i.$$

Axiom 2. Each actor in the society has an initial image of the stratification system, and this image consists only of one social class to which the actor belongs. The images of the stratification system change through interaction with others.

Axiom 3. In each interaction among actors, each actor's image is transformed according to the following postulated process: The actor searches for information as to the class location of the other actor (termed an alter) in an order corresponding to the lexicographic ordering and the search continues until a class distinction is made or all dimensions have been exhausted, whichever comes first.

Axiom 4. The following postulated process transforms the state of the image of any actor as a consequence of any interaction. Let $cl(a)$ be the sampled part of the class location of alter a, when the information search process has been completed in that interaction. Then the following four transition rules determine the transformation of the image state:

  1. If $cl(a)$ is represented in the image that is already held by the actor, no change occurs.
  2. If $cl(a)$ is above the highest class in the image, then $cl(a)$ becomes the new highest class.
  3. If $cl(a)$ is lower than the lowest class in the image, then $cl(a)$ becomes the new lowest class.
  4. If $cl(a)$ is between some pair of classes in the image, then it is "inserted" between them in the new image.

    Given that there is at least one actor in each class and each actor randomly encounters each other actor in the society, then sooner or later every actor will have a stable image of the system, which will remain unchanged by any further encounters. The last axiom is about the determination of class identification that is also called as "self-location in the stratification system" in Fararo and Kosaka's terminology.

Axiom 5. An actor expresses his/her class identification on the basis of the relative location of his/her affiliation class on his/her own stable image of the stratification system.

3 Derivations of the Model

Findings of the Fararo-Kosaka model derived from the model axioms are examined in detail in a series of articles and books by Fararo and Kosaka. Here, I only state the propositions relevant to the distribution of class identification.

Proposition 1. The number of classes in any stable image is the same for all actors and is given by

$$n = s(r-1) + 1. \tag{1}$$

Proof. In the i-th dimension, an actor belonging to $(k_1, k_2, \cdots, k_s)$ recognizes $r - 1 - k_i$ classes of higher rank and k_i classes of lower rank. Therefore, the number of classes recognized by an actor is $r - 1 - k_i + k_i = r - 1$. Because this applies to all dimensions, we have $s(r-1)$ classes. By adding 1, which is the number of the affiliation class of the actor, we arrive at Equation (1).

Here, let us label the ranks on a stable class image from the top to the bottom as $s(r-1)$, $s(r-1) - 1, \cdots, 2, 1, 0$.

Proposition 2. For an actor who belongs to $(k_1, k_2, \cdots, k_s)$, a rank of the affiliation class on a stable class image, which is assumed to be actor's class identification, denoted by ρ, is given by

$$\rho = \sum_{i=1}^{s} k_i. \tag{2}$$

Proof. The rank ρ is given by the total number of classes recognized to be of lower rank by the actor. As the total number of classes recognized to be of lower rank in the i-th dimension is k_i, the summation of all dimensions yields Equation (2).

According to Fararo and Kosaka (1991), we now introduce the term "chance society" to denote a society in which an equal number of people are assigned to each logically possible class in a stratification system. In addition, to simplify the following description, we assume that one person is assigned to each class. Then, the last proposition is about the distribution of class identification in a "chance society."

Proposition 3. In a "chance society," the distribution of class identification ρ is given by a function $f(\rho)$ such that
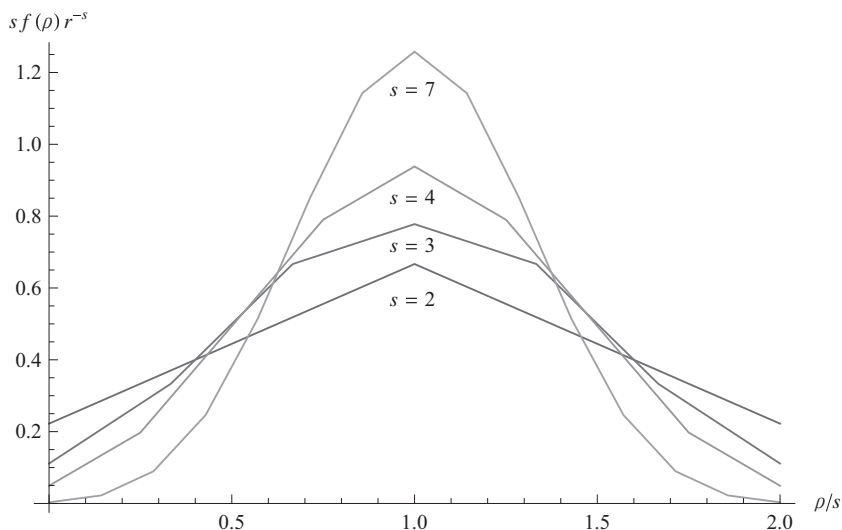

Figure 1: Distribution of class identification in a chance society ($r = 3$)

$$f(\rho) = \sum_{k=0}^{[\rho/r]} (-1)^{k+\rho-rk} \binom{s}{k} \binom{-s}{\rho-rk} \tag{3}$$

$$= \sum_{k=0}^{[\rho/r]} (-1)^k \binom{s}{k} \binom{s+\rho-rk-1}{s-1} \tag{4}$$

where $[a]$ denotes the integer part of a, and $\binom{s}{k}$ denotes a binomial coefficient.

Kosaka and Fararo (1991), Fararo and Kosaka (2003), and Kosaka (2006) present this distribution function in the form of Equation (4) from a textbook on combinatorial mathematics by Niven (1965). On the other hand, Yosano (1996) and Kosaka and Yosano (1998) present this proposition in the form of Equation (3) from the proposition of the convolution of independent discrete uniform distributions (Feller 1957). Given that each dimension of stratification system is independently uniformly distributed over C_i, we get the probability function of class identification by dividing the above equation by r^s. Furthermore, apart from the chance society assumption, according to the central limit theorem, it is concluded that relative class identification ρ/s is asymptotically normally distributed on $N(\mu, \sigma^2/s)$ if $s \to \infty$ given that each dimension is independently and identically distributed with mean μ and variance σ^2 (Yosano 1996; Kosaka and Yosano 1998). Finally, Hamada (2012) proves by using the Lyapounov's central limit theorem that relative class identification is asymptotically normally distributed no matter how each dimension is distributed, as long as each independent distribution of dimension has finite μ, σ^2 and the maximum value of ranks.

These propositions are mathematically true and there are no words to be added to the propositions in terms of mathematics. However, it is worth noting the detailed derivation of Proposition 3, especially for undergraduate students who have a strong desire to learn mathematical sociology and want to completely understand the derivation. Therefore, in the next section, we present the derivation of Proposition 3 in terms of the convolution of sequences.

4 Detailed Derivation of Proposition 3

Let us consider the distribution of class identification in a chance society where one person is assigned to each class.

4. 1 A Case with $r = 2$

First, we shall consider the case in which each dimension of the stratification system consists of two ranks $C_i = \{0, 1\}$. We want to determine the number of s-tuples $(k_1, k_2, \cdots, k_s)$, $k_i \in \{0, 1\}$ that satisfy the constraint condition $\rho = \sum_{i=1}^{s} k_i$. In this case, if ρ of the s dimensions are 1 and the rest are 0, then the condition is satisfied. This leads to "the combination of s objects taken ρ at a time." Hence, the number of actors whose class identification is ρ, denoted by $f(\rho)$, is given by

$$f(\rho) = \binom{s}{\rho} = {}_sC_\rho = \frac{s!}{\rho!(s-\rho)!}. \tag{5}$$

From the binomial theorem,

$$(1+x)^s = \sum_{\rho=0}^{s} \binom{s}{\rho} x^\rho. \tag{6}$$

Table 1　The number of dimensions s and Pascal's triangle ($r = 2$)

s	generating function												
0	$(1+x)^0$						1						
1	$(1+x)^1$					1		1					
2	$(1+x)^2$				1		2		1				
3	$(1+x)^3$			1		3		3		1			
4	$(1+x)^4$		1		4		6		4		1		
5	$(1+x)^5$	1		5		10		10		5		1	
6	$(1+x)^6$	1	6		15		20		15		6		1

This identity indicates that $f(\rho)$ coincides with a coefficient of x^ρ in the binomial theorem. Furthermore, $(1+x)^s$ is the generating function of each row in Pascal's triangle. Hence, the distribution of class identification under the condition of s dimensions is shown in Pascal's triangle (Table 1).

4. 2 Generalization

By analogy with the above analysis using the binomial theorem, it can be predicted that if all dimensions have the same rank r, the number of actors of rank ρ is given by the function $f(\rho)$, which determines the coefficient of x^ρ in a expansion of power of a single-variable polynomial, such that

$$\left(\sum_{k=0}^{r-1} x^k\right)^s = \sum_{\rho=0}^{s(r-1)} f(\rho) x^\rho. \tag{7}$$

Indeed, this is right. To prove this equation, we introduce the concept of generating functions of sequences and their convolution[1].

In general, we denote any infinite sequence as

$$\langle a_n \rangle = \langle a_0, a_1, \cdots, a_n, \cdots \rangle.$$

Let us now introduce a function $a(x)$ in which the coefficient of x^ρ corresponds one to one with the k-th element of the sequence $\langle a_n \rangle$, that is

$$a(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 x^0 + a_1 x^1 + \cdots + a_n x^n + \cdots.$$

This function $a(x)$ is called "the generating function" of the sequence $\langle a_n \rangle$.

Let us introduce another infinite sequence as

$$\langle b_n \rangle = \langle b_0, b_1, \cdots, b_n, \cdots \rangle.$$

The generating function of this sequence $\langle b_n \rangle$ is

$$b(x) = \sum_{k=0}^{\infty} b_k x^k.$$

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1) A detailed explanation on the binomial coefficient, generating function of sequence and convolution is given in Ch.5 of Graham et. al. (1989).

Now, we introduce the specific mathematical operation on sequences called "convolution of sequences" which produces another sequence. Strictly, the convolution of $\langle a_n \rangle$ and $\langle b_n \rangle$, denoted by $\langle a_n \rangle * \langle b_n \rangle$, is defined as

$$\langle a_n \rangle * \langle b_n \rangle = \left\langle \sum_{k=0}^{n} a_k b_{n-k} \right\rangle \tag{8}$$
$$= \langle (a_0 b_0),\ (a_0 b_1 + a_1 b_0),\ (a_0 b_2 + a_1 b_1 + a_2 b_0),\ \cdots,\ \sum_{k=0}^{n} a_k b_{n-k},\ \cdots \rangle.$$

Furthermore, the product of the generating functions $a(x)$ and $b(x)$ is

$$a(x)\, b(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x^n. \tag{9}$$

From Equations (8) and (9), we can see that the convolution of sequences corresponds to the product of the generating functions of these sequences.

Generally, the convolution of s infinite sequences $\langle a_{1n} \rangle$, $\langle a_{2n} \rangle$, $\cdots$, $\langle a_{sn} \rangle$ is defined as

$$\langle a_{1n} \rangle * \langle a_{2n} \rangle * \cdots * \langle a_{sn} \rangle = \left\langle \sum_{k_1 + \cdots + k_s = n} a_{1k_1}\, a_{2k_2} \cdots a_{sk_s} \right\rangle. \tag{10}$$

We now introduce a specific infinite sequence U_r whose first r elements are 1 and the remaining are 0, that is

$$U_r = \langle u_0,\ u_1,\ \cdots,\ u_{r-1},\ u_r,\ \cdots \rangle$$
$$= \langle \underbrace{1,\ 1,\ \cdots,\ 1}_{r \text{ objects}},\ 0,\ 0,\ \cdots \rangle$$

where

$$u_k = \begin{cases} 1\ (k \leq r - 1) \\ 0\ (k > r - 1) \end{cases}$$

We denote the generating function of U_r as $U_r(x)$, that is

$$U_r(x) = \sum_{k=0}^{r-1} x^k.$$

For example, $U_2 = \langle 1,\ 1,\ 0,\ 0,\ \cdots \rangle$ and its generating function is $(1 + x)$.

From the definition of convolution, the ρ−th element of the convolution of s objects of U_r is

$$\sum_{k_1 + \cdots + k_s = \rho} u_{k_1}\, u_{k_2} \cdots u_{k_s}. \tag{11}$$

If at least one k_i exceeds $r - 1$, then the term $u_{k_1} u_{k_2} \cdots u_{k_s}$ is zero. Therefore, Equation (11) yields the number of patterns of s-tuples $(k_1,\ \cdots,\ k_s)$ satisfying the following conditions:

$$\forall i \in \{1,\ \cdots,\ s\},\ k_i \in \{0,\ 1,\ \ldots,\ r-2,\ r-1\},\ \rho = \sum_{i=1}^{s} k_i,$$

and this is equal to the number of actors whose rank is ρ in his/her image of the stratification system in a chance society in the framework of the Fararo-Kosaka model.

Table 2 Outcome of convolution of three objects of U_3

k	0	1	2	3	4	5	6	7
U_3	1	1	1	0	0	0	0	0
U_3	1	1	1	0	0	0	0	0
U_3	1	1	1	0	0	0	0	0
$U_3*U_3*U_3$	1	3	6	7	6	3	1	0

Until now, it is shown that the distribution of class identification in the $s \times r$ stratification system in a chance society is uniquely given by the convolution of s objects of U_r, that is

$$\underbrace{U_r * U_r * \cdots * U_r}_{s \text{ objects}}.$$

Next, we derive the explicit expression of the sequence. To this end, we use the finding that the convolution of sequences corresponds to the product of the generating functions. The generating function that corresponds to convolution of s objects of U_r is given by

$$(U_r(x))^s = \left(\sum_{k=0}^{r-1} x^k \right)^s.$$

We derive the explicit form of $f(\rho)$ by expanding this equation:

$$
\begin{aligned}
\left(\sum_{k=0}^{r-1} x^k \right)^s &= (1 + x + x^2 + x^3 + \cdots + x^{r-1})^s \\
&= \left(\frac{1 - x^r}{1 - x} \right)^s \\
&= (1 - x^r)^s (1 - x)^{-s} \\
&= \left(\sum_{k=0}^{\infty} \binom{s}{k} (-1)^k x^{rk} \right) \left(\sum_{j=0}^{\infty} \binom{-s}{j} (-1)^j x^j \right)
\end{aligned}
\tag{12}
$$

As for the last form of the right-hand side of the equation, both left and right terms are derived by the generalized binomial theorem.

Let us introduce the notations a_{rk}, b_j defined as

$$a_{rk} = \binom{s}{k} (-1)^k, \ b_j = \binom{-s}{j} (-1)^j.$$

Furthermore, b_j can be transformed into the following form:

$$b_j = \binom{-s}{j} (-1)^j = \binom{s + j - 1}{j} = {}_sH_j$$

where ${}_sH_j$ is known as "the repeated combination of s objects taken j with duplication allowed."

By using the notations a_{rk} and b_j, Equation (12) can be transformed into the following:

$$\left(\sum_{k=0}^{r-1} x^k\right)^s = \left(\sum_{k=0}^{\infty} a_{rk} x^{rk}\right)\left(\sum_{j=0}^{\infty} b_j x^j\right). \tag{13}$$

Equation (13) consists of the product of two different generating functions. Comparing with Equation (9), let us examine the coefficient of x^ρ in the expansion of Equation (13). The coefficient of x^ρ is the sum of $a_{rk} b_j$ which satisfies $\rho = rk + j$. Because $j = \rho - rk$ and $j \geq 0$, k moves in the range between 0 and $[\rho /r]$. Hence, the function $f(\rho)$, which is equal to the coefficient of x^ρ, is

$$\begin{aligned}
f(\rho) &= \sum_{k=0}^{[\rho/r]} a_{rk} b_{\rho-rk} = \sum_{k=0}^{[\rho/r]} (-1)^{k+\rho-rk}\binom{s}{k}\binom{-s}{\rho-rk} \\
&= \sum_{k=0}^{[\rho/r]} (-1)^k \binom{s}{k}\binom{s+\rho-rk-1}{\rho-rk} \\
&= \sum_{k=0}^{[\rho/r]} (-1)^k \binom{s}{k}\binom{s+\rho-rk-1}{s-1}.
\end{aligned}$$

Therefore, Equations (3) and (4) in Proposition 3 are derived. The generating function of the distribution of class identification is the following:

$$\begin{aligned}
\left(\sum_{k=0}^{r-1} x^k\right)^s &= \left(\sum_{k=0}^{\infty} a_{rk} x^{rk}\right)\left(\sum_{j=0}^{\infty} b_j x^j\right) \\
&= \sum_{\rho=0}^{s(r-1)} \left(\sum_{k=0}^{[\rho/r]} a_{rk} b_{\rho-rk}\right) x^\rho \\
&= \sum_{\rho=0}^{s(r-1)} \left(\sum_{k=0}^{[\rho/r]} (-1)^{k+\rho-rk}\binom{s}{k}\binom{-s}{\rho-rk}\right) x^\rho.
\end{aligned}$$

5 Conclusion

We presented a detailed derivation of the distribution of class identification in a chance society. We hope that this will aid learners in gaining a deeper understanding of the Fararo-Kosaka model. There are a number of works which apply or extend the model in terms of, for example, images of shape of the stratification (Fararo and Kosaka 1992), social mobility (Watanabe and Doba 1995), cognitive efficientcy (Ishida 2003), reference group (Maeda 2011), as well as theoretical works revisiting the model's derivations by Yosano (1996) and Hamada (2012). Learners interested in this model are advised to review these works next.

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Key Words: class identification, Fararo-Kosaka model, convolution of sequences