

# An Evolutionary Game Analysis of the Boudon-Kosaka Model of Relative Deprivation\*

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## 1. Introduction

In the field of sociology as well as social psychology, the concept of relative deprivation has been one of the key concepts used in the studies of people's attitudes, aspirations, grievances, and collective behaviors such as social movements or revolutions (Gurr 1968). The term relative deprivation was first coined by Stouffer and his colleagues in the classical empirical study *The American Soldier* (Stouffer et al. 1949). Later, Merton and Kitt (1950) summed up their findings and theoretically discussed the relation between relative deprivation and the concept of reference groups.

Crosby (1976) proposed five (effectively six) preconditions of relative deprivation on the basis of a review of theories of relative deprivation mainly by Davis (1959), Runciman (1966), and Gurr (1970). Crosby (1976: 90) states that

*A* is relatively deprived of *X* when (1) *A* lacks *X*, (2) *A* sees that someone else possesses *X*, (3) *A* wants *X*, (4) *A* feels entitled to *X*, (5) *A* think it feasible to obtain *X*, and (6) *A* lacks a sense of personal responsibility for not having *X*.

Crosby's definition is more comprehensive and clear than other definitions, and hence, in my paper I would like to adopt this definition as a verbal definition for my discussion. By highlighting a sense of entitlement and lack of responsibility in the preconditions of relative deprivation, we can see that the feeling of relative deprivation is closely related to a sense of justice, fairness, and equity. Hence, the concept of relative deprivation, mostly along with the concept of reference groups, is crucial not only for studies on people's consciousness and attitudes—wherein these psychical processes are often regarded as the process mediating objective situation to actual behaviour of people—but also for normative studies of the evaluation of the distribution of social resources, that is, distributive justice, and studies on inequality indices (Sen 1973; Yitzhaki 1979).

Boudon (1982) proposed a unique, although primitive, formal model of relative deprivation. Later, Kosaka (1986) elaborated on this model and derived some interesting implications. In this paper, I refer to this model as the Boudon-Kosaka model. This model has a series of followers (Yamaguchi

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**\*Key Words: relative deprivation, evolutionary game, replicator dynamics**

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1998; Reyniers 1998; Hamada 2007), and each of them have revised and expanded the original model in different ways. In this article, I attempt to reformulate the Boudon-Kosaka model within the framework of the evolutionary game theory, specifically, the theory of replicator dynamics. By doing so, I aim to analyze the dynamic process of the model and then clarify the paradoxical relationship between people's rational choices at the microlevel and the emergence of relative deprivation at the macrolevel. Furthermore, I attempt to generalize the Boudon-Kosaka model into the  $n$ -strategy investment game in order to describe more realistic situations such as investment choices in a labor market. By comparing two different operational definitions of relative deprivation in the  $n$ -strategy game, we can evaluate the macro consequences of micro rationality from various perspectives.

In the following section, I present the original model and analyze it. In section 3, I translate the Boudon-Kosaka model into the frame of the replicator dynamics theory and analyze its dynamics. In section 4, I analyze the three-strategy investment game as an exemplar of the generalized  $n$ -strategy investment game. Finally, in section 5, I present the conclusion and mention some future research tasks.

## 2. The Boudon-Kosaka Model

### 2.1 Axioms of the Model

The Boudon-Kosaka model represents a situation where each person in the society interdependently decides between a high-risk-high-return investment and a no-risk-low-return investment. The following are the axioms of the model (Kosaka 1986: 36).

**Axiom 1** *There are  $N$  players in the society.*

**Axiom 2** *Players are offered binary choices.*

*Move 1: Stake  $C_1$  for a possible win of  $B_1$  and*

*Move 2: Stake  $C_2$  for a win of  $B_2$ ,*

*where  $B_1 > C_1$ ,  $B_2 > C_2$ ,  $B_1 > B_2 \geq 0$ ,  $C_1 > C_2 \geq 0$ , and  $B_1 - C_1 > B_2 - C_2$ . The number of players who stake  $C_1$  is denoted by  $x_1$ .*

**Axiom 3** *The numbers of winners of lots  $B_1$  and  $B_2$  are  $n_1$  and  $n_2$ , respectively, assuming that  $n_1 + n_2 = N$ .*

**Axiom 4** *Players are advised to stake  $C_1$  rather than  $C_2$  if the expected net profit of staking  $C_1$  exceeds that of staking  $C_2$ .*

The last condition of Axiom 2, that is,  $B_1 - C_1 > B_2 - C_2$  was not introduced clearly by Kosaka (1986) but was formally introduced by Hamada (2007).  $B_1 - C_1 > B_2 - C_2$  indicates that the profit obtained from the success of Move 1, that is, the win of  $B_1$  by staking  $C_1$ , is always higher than the profit obtained from Move 2, that is, the win of  $B_2$  by staking  $C_1$ ; this seems to be a rather natural assumption of the model.

It is worth noting that when  $B_2 = C_2 = 0$ , Move 2 indicates non participation in the investment game; hence, in this situation, each player has to choose between whether to stake investment  $C_1$  with a contingent return of  $B_1$  or not to participate in the investment. This was a scenario considered by Boudon in his numerical examples (Boudon 1982: 110–6).

Here,  $p_s$  is defined as  $p_s = x_1/N$  and  $\gamma$  as  $\gamma = n_1/N$ , each indicating the proportion of those who choose Move 1 and the winners of lots  $B_1$  in the player group, respectively.  $\gamma$  can also be interpreted as the success rate of staking investment  $C_1$  in the society. We now assume that  $N$  is sufficiently large, and we substitute the notations  $x_1$  and  $n_1$  with  $p_s$  and  $\gamma$ , respectively, in the analysis of the model. Hence, a player’s decision tree, given that the proportion of those who choose Move 1 among the other players is  $p_s - 1/N$ , can be described as Figure 1.

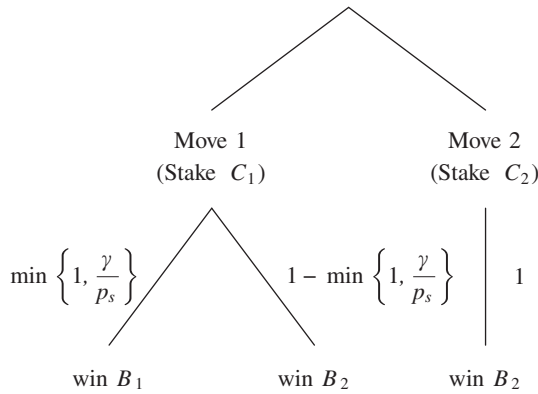


Figure 1: Decision tree for a player

**2. 2 Equilibrium Point**

I introduce  $E(\text{Move 1}; p_s)$  as the expectation of net profit of a player who chooses Move 1, given that the proportion of those who choose Move 1 in the society is  $p_s$ , that is,

$$E(\text{Move 1}; p_s) = (B_1 - C_1) \min \left\{ 1, \frac{\gamma}{p_s} \right\} + (B_2 - C_1) \left( 1 - \min \left\{ 1, \frac{\gamma}{p_s} \right\} \right).$$

The expectation of net profit of a player who chooses Move 2, denoted by  $E(\text{Move 2})$ , is

$$E(\text{Move 2}) = B_2 - C_2.$$

If  $0 < p_s \leq \gamma$ ,  $E(\text{Move 1}; p_s) > E(\text{Move 2})$  holds under all conditions and from the assumption of rationality (Axiom 4), a player is sure to choose Move 1 in the situation. This means that the situation  $p'_s$ , conditioned by  $\gamma > p'_s = p_s - (1/N)$ , can never be an equilibrium. In addition, if  $\gamma = 0$ , then  $\forall p_s \in (0, 1]$ ,  $E(\text{Move 1}; p_s) < E(\text{Move 2})$ , so that every player is sure to choose Move 2. However, this result is inconsequential.

We would now like to examine the cases where  $0 < \gamma \leq p_s$ . Kosaka (1986) insists that an equilibrium point of the proportion of players who choose Move 1 (staking  $C_1$ ), denoted by  $p_s^*$ , is given by the following condition:

$$E(\text{Move 1}; p_s^*) = E(\text{Move 2}). \tag{1}$$

This condition can be solved in terms of  $p_s^*$  as

$$p_s^* = \left( \frac{B_1 - B_2}{C_1 - C_2} \right) \gamma = \frac{B}{C} \gamma, \tag{2}$$

where  $B = B_1 - B_2$ ,  $C = C_1 - C_2$ .  $B/C$  can be regarded as the incremental benefit-cost ratio indicating the ratio of the additional advantage  $B_1 - B_2$  to the additional stake  $C_1 - C_2$ . By Axiom 2,  $B/C > 1$ .

Kosaka also insists that Equation (2) holds as long as  $(B/C)\gamma \leq 1$ , and if  $(B/C)\gamma > 1$ , it becomes

$$p_s^* = 1, \tag{3}$$

which indicates that all of the players in the society choose Move 1.

However, the equilibrium condition of Equation (2) is not directly derived from the axioms of the model, it seems that another assumption of the dynamic process is required for the derivation of the condition.

### 2. 3 Relative Deprivation Rate

In this model, it is assumed that the players who staked  $C_1$  but failed to gain  $B_1$  and eventually won  $B_2$  are put into a position of relative deprivation, since

- (i) they are deprived of  $B_1$ ; (ii) they know that there are some others who adopted the same type of behavior (Move 1) on the same behavioral principle (Axiom 4) and succeeded in obtaining  $B_1$ ;
- (iii) they also want  $B_1$ ; and (iv) they see that they are entitled to gain  $B_1$  by the equal probability to all players who adopted Move 1 (Kosaka 1986: 37).

These statements correspond to Runciman's definitions and Crosby's definitions of (1) through (5). Further, it is reasonable to add the statement, corresponding to Crosby's sixth precondition, that the player's do not need to feel a sense of personal responsibility for losing  $B_1$  because it is a matter of probability. Therefore, the following definition is employed for the rate of relative deprivation  $S$  that is:

$$S = p_s \left( 1 - \min \left\{ 1, \frac{\gamma}{p_s} \right\} \right) = p_s - \min \{ p_s, \gamma \}. \tag{4}$$

If  $p_s < \gamma$  or  $\gamma = 0$ , then  $S = 0$ . Therefore, let us examine the relative deprivation rate under the condition  $0 \leq \gamma \leq p_s$ . According to Equations (2) and (3),  $S$  at the equilibrium point is

$$S = \begin{cases} \left( \frac{B}{C} - 1 \right) \gamma, & 0 \leq \gamma \leq C/B \\ 1 - \gamma, & C/B < \gamma \leq 1 \end{cases} \tag{5}$$

In the above equation,  $B/C - 1$  can be interpreted as the ratio of incremental net profit  $B - C$  to the incremental cost  $C$ .

Figure 2 shows the change in the relative deprivation rate at the equilibrium point by  $\gamma$ —the rate of winning  $B_1$ —according to the levels of the benefit-cost ratio. We can draw several implications from Figure 2 (Kosaka 1986: 41–2). One of the main findings is that the function of deprivation rate is a trigonometric function with the peak at  $\gamma = C/B$ . The increasing slope of the function, as inter-

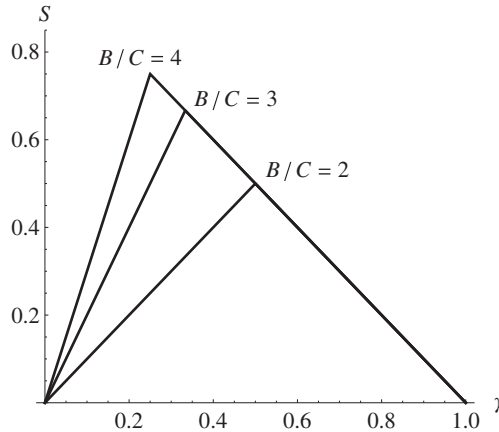


Figure 2: The relative deprivation rate at the equilibrium point on the basis of  $\gamma$  and  $B/C$

preted by Boudon and Kosaka, corresponds to *the American Soldier's* paradox: the units with relatively high promotion opportunities report experiencing relatively higher levels of frustration, while the units with low promotion opportunities tend to be content with such a situation.

### 3. Evolutionary Game Analysis of the Two-strategy Investment Game

#### 3. 1 Description of the Game and Nash Equilibria

In this subsection, using the framework of the Boudon-Kosaka model, I would like to construct a two-player investment game that represents an interdependent decision-making situation, which might lead the players to feel frustrated.

Figure 3 illustrates the structure of the two-player investment game. In the game, it is assumed that  $\min\{1, g/p_i\}$  and  $1 - \min\{1, g/p_j\}$ , which represent the probability of  $i$ 's success and failure, respectively, in winning  $B_i$  by staking  $C_i$ , is determined by the other player  $j$ 's probability of choosing Move 1 as well as by the parameter  $g$ .  $g$  is interpreted as the minimum success probability when the other player  $j$  is certain to choose Move 1, that is,  $p_j = 1$ . Apparently, as  $p_j$  approaches 1,  $\min\{1, g/p_j\}$  decreases from 1 to  $g$  while  $1 - \min\{1, g/p_j\}$  increases from 0 to  $1 - g$ . This means that as one

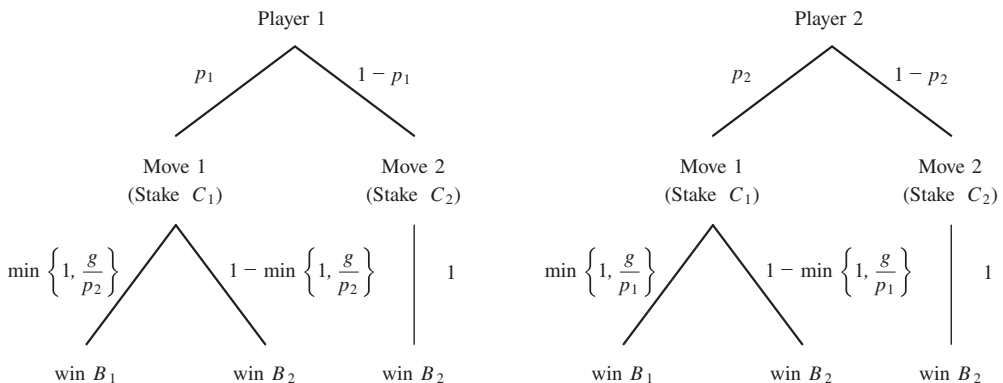


Figure 3: Two-players investment game

player's probability of choosing Move 1 increases, the other player's probability of winning  $B_1$  in the high-risk stake decreases and the probability of the stake's failure increases. In such a situation, player  $i$  chooses a mixed strategy  $(p_i, 1 - p_i) \in \Delta$ , where  $\Delta$  is the mixed strategy set common to players.

In this game, the mixed strategy function of player  $i$ , denoted by  $u_i(p_1, p_2)$ , is

$$u_i(p_1, p_2) = \left( \min \left\{ 1, \frac{g}{p_j} \right\} B - C \right) p_i + B_2 - C_2. \tag{6}$$

Let us denote the best-reply correspondence of player  $i$  to the other player  $j$ 's strategy  $p_j$  by  $\beta_i(p_j)$ . If  $g = 0$ ,  $\beta_i(p_j) = 0$  for all  $p_j \in [0, 1]$ , and consequently, we have only one Nash equilibrium at  $(p_1^*, p_2^*) = (0, 0)$ . If  $g \neq 0$ , it is assumed that  $p_i > 0$  ( $i = 1, 2$ ). Under this condition, every mixed strategy  $p_i$  in the interval  $(0, g]$  is identical in terms of the best-reply correspondence of player  $j$ . Hence, in the following analysis, I let  $g$  represent the interval  $(0, g]$  and assume  $g \leq p_i \leq 1$  ( $i = 1, 2$ ) for simplicity of the description. Under this assumption, the mixed strategy set of this game has to be  $\Delta = \{(p_i, 1 - p_i) \mid p_i \in [g, 1]\}$ . Hence, this game is an anomalous variant of a mixed extended  $2 \times 2$  symmetric game.

If  $0 < g < C/B$ ,  $\beta_i(p_j)$  is

$$\beta_i(p_j) = \begin{cases} g, & p_j > (B/C)g \\ [g, 1], & p_j = (B/C)g \\ 1, & g \leq p_j < (B/C)g \end{cases}$$

and if  $g = C/B$ , it is

$$\beta_i(p_j) = \begin{cases} [g, 1], & p_j = 1 \\ 1, & g \leq p_j < 1. \end{cases}$$

Finally, if  $C/B < g \leq 1$ , then we have  $\beta_i(p_j) = 1$  for all  $p_j \in [g, 1]$ .

From these equations, we can determine the Nash equilibria of this game. Let us denote the set of Nash equilibria by  $\Delta^{NE}$ . If  $0 < g < C/B$ , it is

$$\Delta^{NE} = \{((B/C)g, (B/C)g), (g, 1), (1, g)\},$$

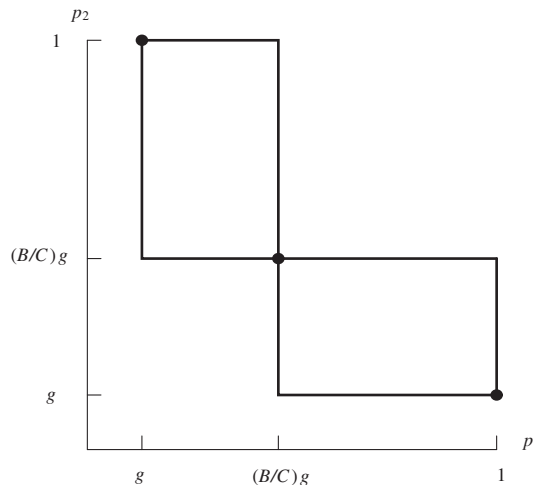


Figure 4: Combination of best-reply correspondences in the two-player investment game (when  $0 < g < C/B$ )

and if  $g = C/B$ ,

$$\Delta^{NE} = \{(1, 1)\} \cup \{(1, p_2) \mid p_2 \in [g, 1)\} \cup \{(p_1, 1) \mid p_1 \in [g, 1)\}.$$

Further, if  $C/B < g \leq 1$ ,  $\Delta^{NE} = \{(1, 1)\}$ . As a result, we find that when  $0 < g < C/B$ , this game is a variant of the chicken game with respect to the payoff structure (see Figure 4).

### 3. 2 The Replicator Dynamics

In this subsection, I would like to introduce the framework of the evolutionary game theory and analyze the replicator dynamics of the game (Taylor and Jonker 1978; Weibull 1995; Hofbauer and Sigmund 1998) in order to figure out the dynamic processes resulting in an equilibrium when all members in the society play the two-player investment game with each other at random.

In this study, I would like to adopt the replicator dynamics as the selection dynamics of the population share of the players who adopt the strategies in the society. The replicator dynamics was first proposed as a biological evolution model, in which the players reproduce offsprings, who then inherit a pure strategy. Hence, this dynamics is better fitted to biological models. However, the replicator dynamics can also be interpreted as a special case of the replication dynamics under the mechanism of a pure imitation process driven by the players' dissatisfaction (Weibull 1995: 152–5). The pure imitation dynamics is a type of simple selection dynamics driven by the players' learning process, which is rather suitable for models of social scientific situations. Hence, I would like to adopt the replicator dynamics as a pure imitation dynamics.<sup>1)</sup>

Here, let us introduce basic notations. Let us focus on symmetric two-player games. The set of pure strategies is denoted by  $K = \{1, 2, \dots, k\}$ , and the mixed strategy set, that is, the mixed strategy simplex, is denoted by  $\Delta = \{\mathbf{x} \in \mathbb{R}_+^k \mid \sum_{i \in K} x_i = 1\}$ . The payoff for strategy  $\mathbf{x} \in \Delta$ , when played against  $\mathbf{y} \in \Delta$ , is denoted as  $u(\mathbf{x}, \mathbf{y})$ .

Suppose that each player in the population is programmed to play a certain pure strategy in the symmetric two-player game considered. Further, suppose that pairs of players are repeatedly drawn at random with equal probability to play the game. Let  $\mathbf{x}(t) = (x_1(t), x_2(t), \dots, x_k(t))$  be a population state where each component  $x_i(t)$  is the population share of players who are programmed to play pure strategy  $i \in K$  in the game at time  $t$ . Hence,  $\mathbf{x}(t)$  is identical with a mixed strategy in the game, that is,  $\mathbf{x}(t) \in \Delta$ .

When the population state is  $\mathbf{x}(t) \in \Delta$ , the expected payoff for a pure strategy  $i$  at a random match is  $u(\mathbf{e}^i, \mathbf{x})$ , where  $\mathbf{e}^i$  is a unit vector whose  $i$ -th component is unity and the others are zero, for example,  $\mathbf{e}^1 = (1, 0, \dots, 0)$ . The population average payoff is

$$u(\mathbf{x}, \mathbf{x}) = \sum_{i \in K} x_i u(\mathbf{e}^i, \mathbf{x}).$$

Then, the replicator dynamics of the game is defined as

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1) It is possible to adopt a more complicated selection dynamics by the learning process, such as trial and error or best responses, instead of the replicator dynamics (Roth and Erev 1995; Erev and Roth 1998; Fudenberg and Levine 1998). However, with respect to an asymptotically stable point, almost similar qualitative tendencies to the replicator dynamics can be obtained by these dynamics.

$$\frac{dx_i}{dt} = [u(\mathbf{e}^i, \mathbf{x}) - u(\mathbf{x}, \mathbf{x})]x_i, \quad i = 1, 2, \dots, k. \quad (7)$$

Consider the replicator dynamics of the the two-player investment game. Let us denote the population share of players who choose Move 1, that is, players who stakes  $C_1$ , in the society by  $p_s$ , and denote the population share of players who choose Move 2 by  $1 - p_s$ . Then, the replicator dynamics of the game is

$$\begin{aligned} \frac{dp_s}{dt} &= [u(1, p_s) - u(p_s, p_s)] p_s \\ &= [u(1, p_s) - u(0, p_s)] (1 - p_s)p_s. \end{aligned} \quad (8)$$

In this evolutionary game situation, an arbitrary player plays the two-player investment game with one of the other players after another in a random sequence. In fact, this process is similar to the situation where the player plays the two-player investment game with the average player who adopts the mixed strategy  $(p_s, 1 - p_s)$ . If the player plays the game with the mixed strategy  $(p_s, 1 - p_s)$ , where  $g \leq p_s$ , the player's success probability of staking  $C_1$  (Move 1) would be  $g/p_s$ . Moreover, there are players who choose Move 1 in the society at the rate of  $p_s$ . Eventually, the success rate of staking  $C_1$  in the society would be  $p_s(g/p_s)$ , and this rate is conceptually equal to  $\gamma$ . Therefore,  $\gamma = g$ . In the following equation,  $g$  is substituted by  $\gamma$  for simplicity. Similarly, the failure rate of staking  $C_1$  in the society would be

$$S = p_s \left( 1 - \min \left\{ 1, \frac{\gamma}{p_s} \right\} \right) = p_s - \min \{ p_s, \gamma \},$$

which is equivalent to the relative deprivation rate, defined in Equation (4), of the original Boudon-Kosaka model.

Since  $u(1, p_s) = E(\text{Move 1}; p_s)$  and  $u(0, p_s) = E(\text{Move 2})$ , the condition for achieving the inner stationary point of the replicator dynamics, that is,  $u(1, p_s) = u(0, p_s)$  is equivalent to condition (1) for achieving the equilibrium point of the original Boudon-Kosaka model. Hence,

$$\begin{aligned} \frac{dp_s}{dt} &= [E(\text{Move 1}; p_s) - E(\text{Move 2})] (1 - p_s)p_s \\ &= \left( \min \left\{ 1, \frac{\gamma}{p_s} \right\} B - C \right) (1 - p_s)p_s. \end{aligned} \quad (9)$$

If  $0 < p_s < \gamma$ , then  $dp_s/dt > 0$ , so that the number of players choosing Move 1 increases until it reaches  $\gamma = p_s$ . If  $\gamma \leq p_s \leq 1$ , the behavior of the dynamics depends on the parameters. In the following, we examine the dynamics in the interval  $[\gamma, 1]$ .

This replicator dynamics is a unidimensional dynamical system. Therefore, it is easy to determine the stationary points of the system and examine their asymptotical stability (Gintis 2000: 175–7). Suppose  $f(p_s) = dp_s/dt$ . The Stationary points  $p_s^*$  are the solutions to the equation  $f(p_s) = 0$ . With respect to stability, if  $f'(p_s^*) < 0$ , then the stationary point  $p_s^*$  is asymptotically stable; meanwhile, if  $f'(p_s^*) > 0$ , then  $p_s^*$  is unstable. If  $f'(p_s^*) = 0$ , then the second-order condition should be checked for judging the stability.

Let us check the stationary points of the dynamics. If  $0 < \gamma < C/B$ , there are two stationary points



in  $[\gamma, 1]$ ,  $p_s^* = 1$  and  $p_s^* = (B/C)\gamma$ . Since  $f'(1) = C - \gamma B > 0$ , the point  $p_s^* = 1$  is unstable. At the same time, the point  $p_s^* = (B/C)\gamma$  is asymptotically stable since  $f'((B/C)\gamma) = \gamma B - C < 0$  (Figure 5). If  $\gamma = C/B$ , there is only one stationary point  $p_s^* = 1$  in  $[\gamma, 1]$ , and from  $f'(1) = 0$ ,  $f''(1) = 2C > 0$  and the fact that the point is at the left end of the domain, we see that the point  $p_s^* = 1$  is asymptotically stable (Figure 6). Finally, if  $C/B < \gamma \leq 1$ , there is only one stationary point  $p_s^* = 1$  in  $[\gamma, 1]$ , and  $p_s^* = 1$  is asymptotically stable since  $f'(1) = C - \gamma B < 0$  (Figure 7).

We can clearly see that  $S$  increases as  $p_s$  increases, and if  $\gamma$  is (nearly) equal to or larger than the incremental cost-benefit ratio  $C/B$ , say, 0.5 as a winning rate along with a double benefit to cost, almost all people would be instigated to stake the high-risk investment, paradoxically, despite the rise in the relative deprivation rate in the society (see Figures 6 and 7). This is a kind of unintended consequence where micro-rationality is aggregated to macro-irrationality (Boudon 1982).

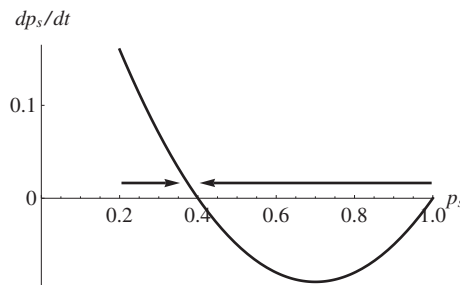


Figure 5: The phase diagram when  $0 < \gamma < C/B$  ( $B=2, C=1, \gamma=0.2$ )

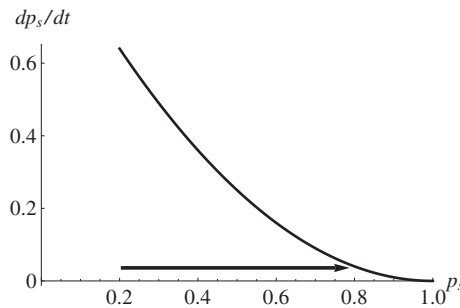


Figure 6: The phase diagram when  $\gamma = C/B$  ( $B=5, C=1, \gamma=0.2$ )

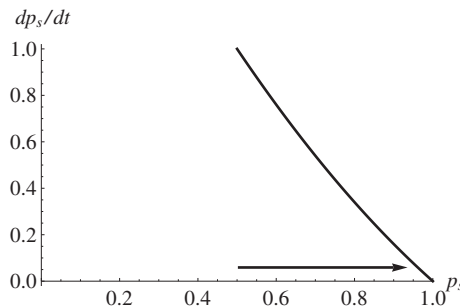


Figure 7: The phase diagram when  $C/B < \gamma \leq 1$  ( $B=5, C=1, \gamma=0.5$ )

## 4. Evolutionary Game Analysis of the Three-strategy Investment Game

### 4.1 Description of the Game

The Boudon-Kosaka model analyzed in the previous section is a two-strategy decision-making model. The model can easily be generalized into the  $n$ -strategy decision-making model in which each investment strategy has a different benefit-cost ratio and different a success rate  $\gamma$ .

This generalized investment game is useful for describing more realistic situations such as strategic behaviors in a labor market, and is also useful for analyzing mechanisms explaining how players' rational choices are aggregated into a macro-status like the distribution of achievement and the degree of relative deprivation in the society.

In this section, I would like to analyze the investment game with three strategies as an exemplar of the generalization of the Boudon-Kosaka model. The decision tree for player  $i$  in this model is illustrated in Figure 8. In this model, I assume that  $B_1 > B_2 > B_3 \geq 0$ ,  $C_1 > C_2 > C_3 \geq 0$ , and  $B_1 - C_1 > B_2 - C_2 > B_3 - C_3 \geq 0$  as a natural extension of the original Boudon-Kosaka model's second axiom. I also assume that  $0 < \gamma_1 \leq p_{j1}$ ,  $0 < \gamma_2 \leq p_{j2}$ , and  $\gamma_1 + \gamma_2 \leq 1$ . Furthermore, in the following analysis, for simplicity of the analysis and the interpretation, I suppose that  $B_3 = C_3 = 0$ . Hence, Move 3 can be interpreted as the "withdrawal strategy" wherein a player does not participate in any investment with no loss and no gain.

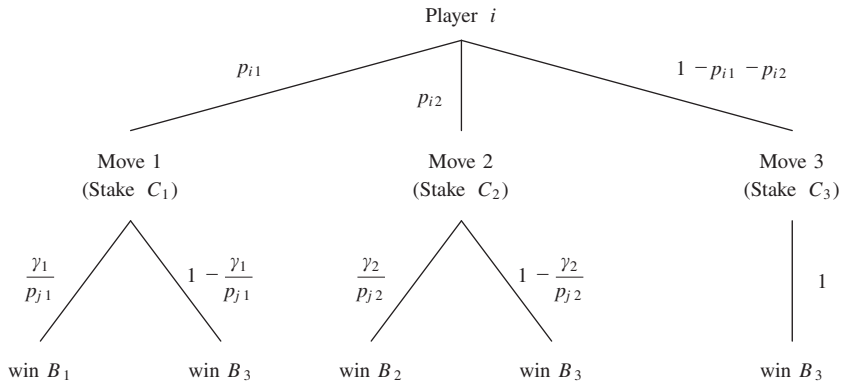


Figure 8: The decision tree for player  $i$  in the three-strategy investment game

### 4.2 The Replicator Dynamics

Let us analyze the replicator dynamics of the game. A mixed strategy of player  $i$  is indicated by a point in the two-dimensional Cartesian coordinates,  $\mathbf{p}_i = (p_{i1}, p_{i2})$ . Further, the expected payoff function of player  $i$  is

$$u_i(\mathbf{p}_1, \mathbf{p}_2) = p_{i1} \left( \frac{\gamma_1}{p_{j1}} B_1 - C_1 \right) + p_{i2} \left( \frac{\gamma_2}{p_{j2}} B_2 - C_2 \right).$$

Let  $\mathbf{p} = (p_1, p_2)$  be the population share of players who choose Move 1 and Move 2 in the society, and let  $\Delta$  be the set of possible population shares, that is,

$$\Delta = \{(p_1, p_2) \in [\gamma_1, 1] \times [\gamma_2, 1] \mid p_1 + p_2 \leq 1\}.$$

The replicator dynamics of the game is represented by the following simultaneous differential equations:

$$\begin{aligned} \frac{dp_1}{dt} &= f(p_1, p_2) = [u(\mathbf{e}^1, \mathbf{p}) - u(\mathbf{p}, \mathbf{p})]p_1, \\ &= (1 - p_1)(B_1\gamma_1 - C_1p_1) - p_1(B_2\gamma_2 - C_2p_2), \end{aligned} \tag{10}$$

$$\begin{aligned} \frac{dp_2}{dt} &= \phi(p_1, p_2) = [u(\mathbf{e}^2, \mathbf{p}) - u(\mathbf{p}, \mathbf{p})]p_2 \\ &= (1 - p_2)(B_2\gamma_2 - C_2p_2) - p_2(B_1\gamma_1 - C_1p_1). \end{aligned} \tag{11}$$

By solving the simultaneous equations  $f(p_1, p_2) = 0$ ,  $\phi(p_1, p_2) = 0$ , we can find two stationary points of the dynamics in the domain  $\Delta$ ,

$$\mathbf{p}^* = (\beta_1, \beta_2), \quad \mathbf{q}^* = (\pi, 1 - \pi),$$

where,  $\beta_k = (B_k/C_k)\gamma_k$  ( $k = 1, 2$ ), and

$$\pi = \frac{C_1 - C_2 + B_1\gamma_1 + B_2\gamma_2 - \sqrt{4(C_1 - C_2)B_2\gamma_2 + (B_1\gamma_1 + B_2\gamma_2 - C_1 + C_2)^2}}{2(C_1 - C_2)}$$

I conduct here a local stability analysis based on linearization around each stationary point (Gintis 2000: 178–181). Let  $f_x(x^*, y^*)$  be the partial derivative of  $f$  at  $(x^*, y^*)$  with respect to  $x$ , then the Jacobian matrix of the dynamical system at stationary points  $\mathbf{p}^*$  and  $\mathbf{q}^*$  are

$$\begin{aligned} A_1 &= \begin{bmatrix} f_{p_1}(\beta_1, \beta_2) & f_{p_2}(\beta_1, \beta_2) \\ \phi_{p_1}(\beta_1, \beta_2) & \phi_{p_2}(\beta_1, \beta_2) \end{bmatrix} = \begin{bmatrix} B_1\gamma_1 - C_1 & (B_1C_2\gamma_1)/C_1 \\ (B_2C_1\gamma_2)/C_2 & B_2\gamma_2 - C_2 \end{bmatrix}, \\ A_2 &= \begin{bmatrix} f_{p_1}(\pi, 1 - \pi) & f_{p_2}(\pi, 1 - \pi) \\ \phi_{p_1}(\pi, 1 - \pi) & \phi_{p_2}(\pi, 1 - \pi) \end{bmatrix} \\ &= \begin{bmatrix} -(1 - 2\pi)C_1 + (1 - \pi)C_2 - B_1\gamma_1 - B_2\gamma_2 & \pi C_2 \\ (1 - \pi)C_1 & \pi C_1 + (1 - 2\pi)C_2 - B_1\gamma_1 - B_2\gamma_2 \end{bmatrix}. \end{aligned}$$

Let us examine the eigenvalues of each Jacobian matrix. As for  $A_1$ , there are two different real eigenvalues,  $\lambda_{11}$  and  $\lambda_{12}$ , and if  $1 > \beta_1 + \beta_2$ , then  $0 > \lambda_{11} > \lambda_{12}$ , indicating that the stationary point  $\mathbf{p}^*$  is asymptotically stable.<sup>2)</sup> Similarly, there are two different real eigenvalues of  $A_2$ ,  $\lambda_{21}$  and  $\lambda_{22}$ . If  $1 > \beta_1 + \beta_2$ , then  $\lambda_{21} > 0$ ,  $\lambda_{22} < 0$ , indicating that the stationary point  $\mathbf{q}^*$  is a saddle point. In contrast, if  $1 < \beta_1 + \beta_2$ , then  $0 > \lambda_{21} > \lambda_{22}$ , denoting that point  $\mathbf{q}^*$  is stable.

To sum up, if  $1 > \beta_1 + \beta_2$ , there is one inner stationary point  $\mathbf{p}^*$  that is asymptotically stable and there is one saddle point  $\mathbf{q}^*$  on the edge of  $\Delta$ , the line defined as  $p_2 = 1 - p_1$ . On the contrary, if  $1 < \beta_1 + \beta_2$ , there is only one stable point  $\mathbf{q}^*$ . Figure 9 shows the stationary points and the vector field of the replicator dynamics when  $1 > \beta_1 + \beta_2$ . If  $1 > \beta_1 + \beta_2$ , the asymptotically stable point  $(\beta_1, \beta_2)$  can be set anywhere in the domain  $\Delta$  depending on the values of the parameters  $B_k$ ,  $C_k$ , and  $\gamma_k$  ( $k = 1, 2$ ). With respect to the success rates in the society  $\gamma_k$ , as  $\gamma_k$  increases, point  $\mathbf{p}^*$  approaches point  $\mathbf{q}^*$ , and

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2) If  $1 = \beta_1 + \beta_2$ , then  $0 = \lambda_{11} > \lambda_{12}$ , such that it is unable to judge the stability by the local stability analysis.

finally, if  $\beta_1 + \beta_2 = 1$ , then  $\mathbf{p}^*$  is equal to  $\mathbf{q}^*$ .

It is worth noting that the stationary point  $\mathbf{q}^*$  is equal to the only asymptotically stable point of the two-strategy investment game that is constructed by excluding Move 3, the withdrawal strategy, from the three-strategy investment game illustrated in Figure 8. Hence, when  $1 > \beta_1 + \beta_2$ , the trajectory from  $\mathbf{q}^*$  to  $\mathbf{p}^*$  on the unstable manifold of the saddle point  $\mathbf{q}^*$  indicates the transition from a situation in which all of the players participate in staking either  $C_1$  (Move 1) or  $C_2$  (Move 2) to a situation in which some players change their strategy and withdraw from any investment activities (Move 3).

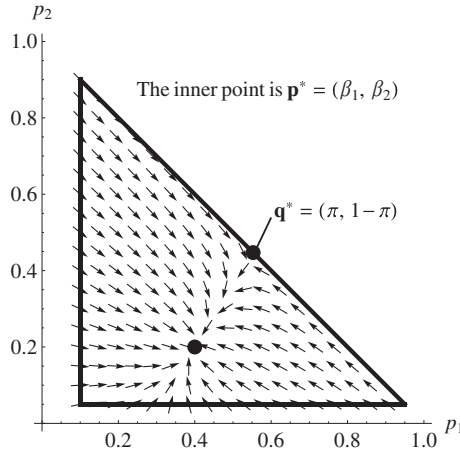


Figure 9: The vector field of the replicator dynamics of the three-strategy investment game ( $B_1=8, B_2=4, C_1=2, C_2=1, \gamma_1=0.1, \gamma_2=0.05$ )

### 4. 3 Evaluation of the Relative Deprivation

When we suppose that people who are relatively deprived by not having what they want and feel entitled to it are players who staked some costs but failed to gain the corresponding benefits in the same manner as the original Boudon-Kosaka model, we have two types of relative deprivation rates according to the strategies, Move 1 and Move 2. The rates of relative deprivation by Move 1 and Move 2 are

$$S_1 = p_1 \left( 1 - \frac{\gamma_1}{p_1} \right) = p_1 - \gamma_1 \quad \text{and} \quad S_2 = p_2 \left( 1 - \frac{\gamma_2}{p_2} \right) = p_2 - \gamma_2, \text{ respectively.}$$

Given that both  $\gamma_1$  and  $\gamma_2$  are constants,  $S_1$  and  $S_2$  increase simply as  $p_1$  and  $p_2$  increase. On the contrary, when  $p_1$  and  $p_2$  decrease as in the case with the trajectory from the saddle point  $\mathbf{q}^*$  to the asymptotically stable point  $\mathbf{p}^*$ ,  $S_1$  and  $S_2$  decrease. This finding can be interpreted as follows: withdrawal from investment by a proportion of the players motivated by rationality improves social welfare in terms of the diminution of the relative deprivation.

However, the relative deprivation rate  $S_1$  and  $S_2$  are derived from one possible operational definition of the degree of relative deprivation in a society; it is possible to consider another type of operational definition that takes distribution of achievement into account. Here, I would like to introduce Yitzhaki's conception of relative deprivation as one possible alternative definition (Yitzhaki 1979).

Yitzhaki (1979) regards the reference group with which people compare themselves as the entire society and regards income as the object of relative deprivation. He defines the relative deprivation of a

person as the sum of the deprivation inherent in all units of income that he/she is deprived of, and the degree of relative deprivation in the society as the average individual deprivation. Let  $D$  be Yitzhaki's relative deprivation index. Yitzhaki formulates the equation  $D = \mu G$ , where  $\mu$  is the average income and  $G$  is the Gini coefficient of income distribution in the society. Thus, Yitzhaki's  $D$  is closely related to the concept of inequality of income distribution.

In the three-strategy investment game, there are five different achievements after the one-shot game  $\mathbf{y} = (B_1 - C_1, -C_1, B_2 - C_2, -C_2, 0)$  with the proportion of population  $\mathbf{r} = (\gamma_1, p_1 - \gamma_1, \gamma_2, p_2 - \gamma_2, 1 - p_1 - p_2)$ . In this situation, Yitzhaki's  $D$  can be represented as the function of a population share of players who adopt each strategies  $\mathbf{p} = (p_1, p_2)$ , that is

$$D(p_1, p_2) = \frac{\sum_i \sum_j r_i r_j |y_i - y_j|}{2},$$

where  $r_i$  and  $y_i$  is the  $i$ -th component of vectors  $\mathbf{r}$  and  $\mathbf{y}$ , respectively. Solving the maximization problem of  $D$  with respect to  $\mathbf{p}$ , we find that the maximizing point is  $(1/2 + \gamma_1, \gamma_2)$ .

It is interesting to consider the behavior of  $D$  on the trajectory from the saddle point  $\mathbf{q}^*$  to the asymptotically stable point  $\mathbf{p}^*$ , when  $1 > \beta_1 + \beta_2$ . We can see that under a certain condition,  $D$  increases when a population share of choosing strategies changes from  $\mathbf{q}^*$  to  $\mathbf{p}^*$  (see Appendix for details). Figure 10 shows a situation where  $D$  increases temporarily as the number of players who withdraw from investment increases. Thus, with respect to Yitzhaki's  $D$ , withdrawal from investment by some players somewhat worsens social welfare under a certain condition.

As described above, the evaluation of the trajectory from  $\mathbf{q}^*$  to  $\mathbf{p}^*$  is quite interesting for examining the different aspects of relative deprivation. A player who rationally withdraws from investment in the three-strategy game can be interpreted as a discouraged worker in a labor market. Some players withdrawing from investment decreases the relative deprivation rates  $S_1$  and  $S_2$ , and this corresponds to the fact that an increase in the number of discouraged workers decreases the unemployment rate in a labor market. However, on the other hand, under a certain condition, some players withdrawing from investment temporarily increases the societal degree of relative deprivation of income  $D$ , and this corresponds to the fact that an increase in the number of discouraged workers often results in an increase in

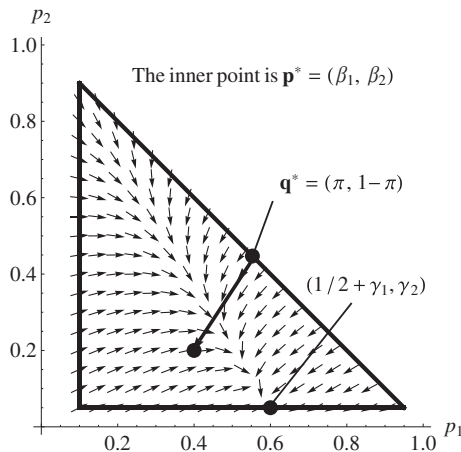


Figure 10: The gradient field of Yitzhaki's relative deprivation index  $D$  ( $B_1=8, B_2=4, C_1=2, C_1=1, \gamma_1=0.1, \gamma_2=0.05$ )

the income inequality. These two types of definitions reflect the differences in the aspects of relative deprivation such as the domain of reference groups and the object of deprivation, and the evaluation of change in social situation resulting from microlevel rationality can vary according to the definitions of relative deprivation.

### 5. Conclusion and Future Tasks

In the preceding sections, I reformulated the Boudon-Kosaka model within the framework of replicator dynamics and analyzed the dynamics of the model. Consequently, I clearly showed the dynamics under which an unintended consequence of rational choice occurs. For example, under a certain condition of the success rate of investment and the cost-benefit ratio, almost all people would be instigated to stake the high-risk investment, despite the rise in the relative deprivation rate in the society. Such dynamics could not be analyzed thoroughly in Boudon and Kosaka’s previous work.

Moreover, I analyzed the replicator dynamics of the three-strategy investment game as an exemplar of the generalized  $n$ -strategy investment game and discussed the differences in the definitions of relative deprivation. In the future, I would like to conduct an in-depth analysis how different relative deprivation indices behave in the  $n$ -strategy investment game; this analysis will probably contribute to an in-depth understanding of the relationship between micro-rationality and the normative evaluations of social situations.

Another future task is to link the analysis of the dynamics that I conducted in this paper and the comparative statics analysis of an equilibrium point of the model. Kosaka (1986) conducted a comparative statics analysis regarding the success rate  $\gamma$  and the benefit-cost ratio as exogenous parameters. By combining a dynamics analysis and a comparative statics analysis, we can resolve a mechanism design problem that addresses the issue of reducing the degree of relative deprivation at the societal level on the basis of people’s rationality at the microlevel.

### Appendix

Let us suppose that  $1 > \beta_1 + \beta_2$ . I present the proposition that under a certain condition, the societal relative deprivation index  $D$  increases when a population share of players who choose the strategies changes from the saddle point  $\mathbf{q}^*$  to the asymptotically stable point  $\mathbf{p}^*$ .

Let  $\nabla f$  be the gradient of  $f$  and let  $\mathbf{u} = (u_1, u_2)$  be the unit vector toward  $(\beta_1 - \pi, \beta_2 - (1 - \pi))$ , that is,

$$\mathbf{u} = \frac{(\beta_1 - \pi, \beta_2 - (1 - \pi))}{\|(\beta_1 - \pi, \beta_2 - (1 - \pi))\|},$$

where  $\|\mathbf{x}\|$  is the Euclidean norm of a vector  $\mathbf{x}$ . Further, the directional derivative of  $D(p_1, p_2)$  at  $(\pi, 1 - \pi)$  in the direction  $(\beta_1 - \pi, \beta_2 - (1 - \pi))$  is defined as

$$\begin{aligned} \nabla_{\mathbf{u}} D(\pi, 1 - \pi) &= \nabla D(\pi, 1 - \pi) \cdot \mathbf{u} = D_{p_1}(\pi, 1 - \pi)u_1 + D_{p_2}(\pi, 1 - \pi)u_2 \\ &= \frac{(\beta_1 - \pi)\{C_1(1 - 2\pi + 2\gamma_1) + 2C_2(\gamma_2 - 1 + \pi)\}}{\sqrt{|\beta_1 - \pi|^2 + |\beta_2 - (1 - \pi)|^2}} \end{aligned}$$

$$+ \frac{(\beta_2 - (1 - \pi))C_2(2\gamma_1 + 2\gamma_2 - 1)}{\sqrt{|\beta_1 - \pi|^2 + |\beta_2 - (1 - \pi)|^2}}.$$

When  $\nabla_{\mathbf{u}}D > 0$  is satisfied, we can say that the proposition is true. A possible sufficient condition of  $\nabla_{\mathbf{u}}D > 0$  simultaneously satisfies the following five conditions:

- (i)  $\pi > \beta_1$ ,      (ii)  $1 - \pi > \beta_2$ ,      (iii)  $1 > 1/2 + \gamma_1 + \gamma_2$ ,  
 (iv)  $\pi > \gamma_1 + 1/2$ ,      (v)  $1 - \pi > \gamma_2$ .

These conditions pertain to the relative positions in the domain  $\Delta$  among  $\mathbf{p}^*$  and  $\mathbf{q}^*$  and  $(1/2 + \gamma_1, \gamma_2)$ , the maximization point of  $D$ . Condition (i) and (ii) call for the configuration that  $\mathbf{p}^*$  is on the left below of  $\mathbf{q}^*$ , condition (iii) simply calls for  $(1/2 + \gamma_1, \gamma_2)$  being in the domain, and conditions (iv) and (v) call for the configuration that  $\mathbf{q}^*$  lies on the upper right of  $(1/2 + \gamma_1, \gamma_2)$ . However, this is just a possible sufficient condition, and there should be many meaningful sufficient conditions depending on the parametric values. For example, we can observe many situations, where conditions (i), (ii), (iii), (v) are satisfied simultaneously but only condition (iv) is not satisfied, satisfy  $\nabla_{\mathbf{u}}D > 0$  (see Figure 10).

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## An Evolutionary Game Analysis of the Boudon-Kosaka Model of Relative Deprivation

### ABSTRACT

In this paper, I attempt to reformulate the Boudon-Kosaka model of relative deprivation within the framework of the replicator dynamics theory, in order to analyze a dynamic process of the model and clarify the micro-macro linkage relevant to the emergence of relative deprivation. The analysis reveals that under certain conditions, nearly all people would be instigated to stake high-risk investments, paradoxically, despite the rise in the relative deprivation rate in society.

Furthermore, I attempt to generalize the Boudon-Kosaka model into the  $n$ -strategy investment game in order to describe more realistic situations. I analyze the replicator dynamics of the three-strategy game as an exemplar, and discuss the difference in the definitions of relative deprivation.

**Key Words:** relative deprivation, evolutionary game, replicator dynamics